

Crash Course: SageMath for Post-Quantum Cryptography

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- + Free, open source, made by volunteers. (**You** can help!)

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- + Tons of functionality, easy to use.
- + Free, open source, made by volunteers. (**You** can help!)
- Sometimes buggy, and sometimes painfully slow.

Note to Python veterans



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- ▶ .sage files are treated as Sage code.
- ▶ .py files are treated as Python code.

Example

```
user@machine:~$ sage
```

```
+-----+  
| SageMath version 9.6, Release Date: 2022-05-15      |  
| Using Python 3.10.5. Type "help()" for help.        |  
+-----+
```

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sage: a^1336                                         # Fermat primality test
973                                                  #  $\neq 1 \implies$  not prime
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sage: is_prime(1337)                                  # Sage agrees
False
```


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973                                                  # ≠ 1 ⇒ not prime
sage: is_prime(1337)                                  # Sage agrees
False
sage: factor(1337)                                    # ...and factors it
7 * 191
```

Parents (1)

`Mod(42, 1337)` showcases a **very important concept**:

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```

```
sage: a
```

```
42
```

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Ring of integers modulo 1337
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sage: a
42
sage: a.parent()
Ring of integers modulo 1337
sage: parent(42)
Integer Ring
sage: parent(42/5)
Rational Field
```

Parents (2)

The **parent determines** the behavior of **many operations**:

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```
sage: -(42)                #  $\mathbb{Z}$ 
-42
sage: -Mod(42, 1000)      #  $\mathbb{Z}/1000$ 
958
sage: -Mod(42, 1337)      #  $\mathbb{Z}/1337$ 
1295
```


Parents (3)

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```
sage: Mod(42, 1000) * 77
234
sage: 5042 == Mod(42, 1000)
True
```

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sage: Mod(42, 1000) * 77
234
sage: 5042 == Mod(42, 1000)
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```

This example does not work:

```
sage: Mod(42, 1000) + 1/7
# ...
TypeError: unsupported operand parent(s) for +:
  'Ring of integers modulo 1000' and 'Rational Field'
```

Parents (4)

Parents are **objects as well** and can be **constructed explicitly**:

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Parents are **objects as well** and can be **constructed explicitly**:

```
sage: R = Zmod(1337)
sage: R
Ring of integers modulo 1337
sage: type(R)
<class 'sage.rings.finite_rings.integer_mod_ring
        .IntegerModRing_generic_with_category'>
```

Parents (5)

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sage: R = Zmod(1337)
```

```
sage: a = 1379
sage: a.parent()
Integer Ring
sage: b = R(a)
sage: b
42
```


Parents (5)

We can **explicitly convert** objects to a **particular parent**:

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sage: R = Zmod(1337)
```

```
sage: a = 1379
sage: a.parent()
Integer Ring
sage: b = R(a)
sage: b
42
```

```
sage: c = 42/5
sage: c.parent()
Rational Field
sage: R(c)
1078
```

Parents (summary)

1. In Sage, *every object* remembers its parent structure.
2. The parent determines the behavior of many operations.
3. Sage automatically coerces between parents in many cases.
4. Parents are normal objects, can be constructed explicitly.
5. We can explicitly convert objects to a particular parent.

Operators

- ▶ $+$, $-$, $*$ all do the obvious thing.

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```
sage: 5 / 3
5/3
sage: 5 // 3      # rounding down
1
```

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5/3
sage: 5 // 3      # rounding down
1
```

- ▶ $^$ is exponentiation (*not* bitwise XOR).
- ▶ \sim is multiplicative inverse (*not* bitwise NOT):

```
sage: ~7
1/7
sage: ~Mod(65537, 2678871240)
444196313
sage: 65537 * 444196313 % 2678871240
1
```

Control structures (1)

...are all inherited from Python:

- ▶ Conditionals:

```
if condition1:  
    # do something  
elif condition2:  
    # do a different thing  
else:  
    # do yet another different thing
```


Control structures (1)

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```
if condition1:  
    # do something  
elif condition2:  
    # do a different thing  
else:  
    # do yet another different thing
```

► Loops:

```
while condition:  
    # do something
```

```
for thing in sequence:  
    # process the thing
```

Control structures (2)

- ▶ Defining and calling a function:

```
def fun(x,y):  
    # do some things with x and y, giving z  
    return z  
  
result = fun("Hello", 5)
```

Control structures (2)

- ▶ Defining and calling a function:

```
def fun(x,y):  
    # do some things with x and y, giving z  
    return z  
  
result = fun("Hello", 5)
```

- ▶ *List comprehensions:*

```
sage: xs = [n*(n+1)//2 for n in range(10)]  
sage: xs  
[0, 1, 3, 6, 10, 15, 21, 28, 36, 45]
```

Polynomials (1)

To construct $R := F[x]$, we can use the syntax:

```
sage: R.<x> = F[]
```

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```
sage: R.<x> = F[]
```

This defines the *generator* x and its parent R . Example:

```
sage: R.<x> = Zmod(101)[]
sage: R
Univariate Polynomial Ring in x over Ring of integers modulo 101
sage: x
x
sage: x.parent() is R
True
```

Polynomials (2)

The resulting variable x can be used to **create polynomials**:

```
sage: R.<x> = Zmod(101)[]
sage: f = x^5 - 3*x^2 + x + 7          # <-----
sage: f.parent()
Univariate Polynomial Ring in x over Ring of integers modulo 101
sage: f
x^5 + 98*x^2 + x + 7
sage: f.degree()                      # degree
5
sage: f(42)                            # evaluation
69
sage: f(42).parent()
Ring of integers modulo 101
sage: f % (x^2-44)                     # and +, *, //, etc.
18*x + 77
```

Polynomials (3)

We can **change the base ring** of many Sage objects:

```
sage: R.<x> = Zmod(101)[  
sage: f = x^5 - 3*x^2 + x + 7  
sage: g = f.change_ring(ZZ)  
sage: g  
x^5 + 98*x^2 + x + 7  
sage: g.parent()  
Univariate Polynomial Ring in x over Integer Ring  
sage: g(42)  
130864153  
sage: g(42) % 101  
69
```

Indexing, iteration, etc.

Most Sage objects support conversions to [lists](#), [indexing](#), etc.:

```
sage: R.<x> = Zmod(101)[  
sage: f = R([7, 1, -3, 0, 0, 1])  
sage: f  
x^5 + 98*x^2 + x + 7  
sage: list(f)  
[7, 1, 98, 0, 0, 1]  
sage: f[2]  
98  
sage: f[9999]  
0  
sage: for c in f: print(c)  
7  
1  
98  
0  
0  
1
```


Quotients (1)

- ▶ Quotients of parents are simply parents again.
- ▶ We've already seen an example:

```
sage: ZZ.quotient(13)
Ring of integers modulo 13
sage: ZZ.quotient(13) is Zmod(13)
True
```

Quotients (2)

- ▶ Example: $\mathbb{Z}[i] \cong \mathbb{Z}[x]/(x^2 + 1)$.

```
sage: P.<x> = ZZ[]
sage: R.<i> = P.quotient(x^2+1)
sage: i^2
-1
sage: i.parent()
Univariate Quotient Polynomial Ring in i
over Integer Ring with modulus x^2 + 1
```

Quotients (2)

- ▶ Example: $\mathbb{Z}[i] \cong \mathbb{Z}[x]/(x^2 + 1)$.

```
sage: P.<x> = ZZ[]
sage: R.<i> = P.quotient(x^2+1)
sage: i^2
-1
sage: i.parent()
Univariate Quotient Polynomial Ring in i
over Integer Ring with modulus x^2 + 1
```

- ▶ Projection and lifting:

```
sage: R(x^3+x^2+1)
-i
sage: (i+1).lift()
x + 1
sage: R(x^3+x^2+1).lift()
-x
```

Finite fields

...are constructed using GF:

```
sage: GF(7)
Finite Field of size 7
```

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```
sage: GF(7)
Finite Field of size 7
```

```
sage: GF(49)
Finite Field in z2 of size 7^2
sage: GF(49).modulus()
x^2 + 6*x + 3
sage: F49.<t> = GF(49)
sage: t^2 + 6*t + 3
0
sage: a = F49.random_element()
sage: a
2*t + 5
sage: a in GF(7)
False
sage: a^8 in GF(7)
True
```

Linear algebra (1)

Matrices and vectors are well-supported:

```
sage: A = matrix([[1,2],[3,4]])
sage: A
[1 2]
[3 4]
sage: v = vector([5,6])
sage: v
(5, 6)
sage: A * v
(17, 39)
```

Vectors are truly **one-dimensional**; can become row or column:

```
sage: v * A
(23, 34)
sage: A.transpose() * v
(23, 34)
```

Linear algebra (2)

Solving **linear systems** over arbitrary fields is straightforward:

```
sage: A = matrix(GF(71), [[1,4,7], [9,1,3]])
sage: y = vector(GF(71), [5,5])
sage: x = A.solve_right(y)           # particular solution
sage: x
(41, 62, 0)
sage: A*x
(5, 5)
sage: ker = A.right_kernel()        # the kernel
sage: ker
Vector space of degree 3 and dimension 1
                                over Finite Field of size 71

Basis matrix:
[ 1 12 64]
sage: ker.basis()
[(1, 12, 64)]
sage: A * vector([1, 12, 64])
(0, 0)
```

Lattices

...are usually best represented as **row matrices over \mathbb{Z}** in Sage.

Example:

```
sage: L = matrix(ZZ, [[21621621617, 1], [99999999977, 0]])
sage: L
[21621621617      1]
[99999999977      0]
sage: R = L.LLL()
sage: R
[      13      37]
[-2405721711 845253590]
sage: R[0, :]
[13 37]
```

(There also exists an `IntegralLattice` class, but I almost never use it.)

Multivariate polynomials (1)

Creating **multivariate polynomial rings** works exactly as before:

```
sage: R.<x,y,z> = QQ[]  
sage: f = x^2 + y^2 - z^2  
sage: f(3,4,5)  
0
```

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0
```

To **access specific coefficients**, we can **index with monomials**:

```
sage: g = x^2 + x*y + 4/5*z^2 - 4/3*z
sage: g[z^2]
4/5
sage: g[x*y]
1
sage: g[y]
0
```

Multivariate polynomials (2)

We can use [many variables](#):

```
sage: R = PolynomialRing(GF(97), 'v', 100)
sage: R
Multivariate Polynomial Ring in v0, v1, v2, ..., v99
      over Finite Field of size 97

sage: R.gens()
(v0, v1, v2, v3, v4, v5, ..., v96, v97, v98, v99)
sage: f = R.random_element()
sage: f
-5*v27*v52 + 39*v2*v55 + 39*v60*v73 - 47*v13*v94
sage: f.monomials()
[v27*v52, v2*v55, v60*v73, v13*v94]
```

Multivariate polynomials (3)

Powerful **equation-solving** tools are available:

```
sage: R.<x,y,z> = QQ[]
sage: I = R.ideal([x^2+y^2-z^2, z-y-1, y-x-1])
sage: I.groebner_basis()
[z^2 - 6*z + 5, x - z + 2, y - z + 1]
sage: I.elimination_ideal([y,z])
Ideal (x^2 - 2*x - 3) of Multivariate Polynomial Ring
                                in x, y, z over Rational Field
sage: factor(x^2 - 2*x - 3)
(x - 3) * (x + 1)
sage: I.variety()
[{z: 5, y: 4, x: 3}, {z: 1, y: 0, x: -1}]
```

Elliptic curves & isogenies

Wait for Thursday! 😊

Cryptographic building blocks (1)

Crucially: **Hashing**. Many functions are available:

```
sage: import hashlib
sage: hashlib.algorithms_guaranteed
{'md5', 'sha1', 'sha224', 'sha256', 'sha384', 'sha512',
 'sha3_224', 'sha3_256', 'sha3_384', 'sha3_512',
 'shake_128', 'shake_256', 'blake2b', 'blake2s'}
```

Usage:

```
sage: hashlib.shake_128(b'Hi!').digest(16)
b't<\xea\xa5\xa7\x88\xd7\xed]\t\xc6\xf1z\xc4e\x96'
sage: list(hashlib.shake_128(b'Hi!').digest(16))
[116, 60, 234, 165, 167, 136, 215, 237, 93, 9, 198,
 241, 122, 196, 101, 150]
```

Cryptographic building blocks (2)

Other helpful tools come from *PyCryptodome*:

```
sage: from Crypto.Cipher import AES
sage: zeroes = bytes([0] * 16)
sage: aes = AES.new(zeroes, AES.MODE_ECB)
sage: aes.encrypt(zeroes).hex()
66e94bd4ef8a2c3b884cfa59ca342b2e
```

Help, I'm lost!

- ▶ Vast [documentation online](#) including [plenty of examples](#).
(In some cases, the documentation is outdated, incomplete, unhelpful, or wrong.)
- ▶ Use `?` in a Sage shell to quickly access the documentation:

```
sage: R.<x> = Zmod(257)[]
sage: R.random_element?
Signature:
    R.random_element(degree=(-1, 2), *args, **kwds)
Docstring:
    Return a random polynomial of given degree
    or with given degree bounds.
# ...
```

- ▶ Feel free to [ask!](#)

Some exercises

https://yx7.cc/docs/sage/sagepqc_taipei_exercises.pdf