Multivariate Quadratic Public-Key Cryptography Part 1: Basics

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Multivariate Cryptography

MPKC: Multivariate (Quadratic) Public Key Cryptosystem
Public Key: System of nonlinear multivariate equations

$$p^{(1)}(x_1,\ldots,x_n) = \sum_{i=1}^{n}\sum_{j=i}^{n} p_{ij}^{(1)}\cdot x_i x_j \quad + \quad \sum_{i=1}^{n} p_i^{(1)}\cdot x_i \left(+p_0^{(1)}\right)$$

$$p^{(2)}(x_1,\ldots,x_n) = \sum_{i=1}^{n}\sum_{j=i}^{n} p_{ij}^{(2)}\cdot x_i x_j \quad + \quad \sum_{i=1}^{n} p_i^{(2)}\cdot x_i \left(+p_0^{(2)}\right)$$

$$\vdots$$

$$p^{(m)}(x_1,\ldots,x_n) = \sum_{i=1}^{n}\sum_{j=i}^{n} p_{ij}^{(m)}\cdot x_i x_j \quad + \quad \sum_{i=1}^{n} p_i^{(m)}\cdot x_i \left(+p_0^{(m)}\right)$$

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Public Key size $= m\binom{n+d}{d}$ at degree d, hence usually $d = 2$.

Security

The security of multivariate schemes is based on the

Problem MQ: Given m multivariate quadratic polynomials $p^{(1)}(\mathbf{x}), \ldots, p^{(m)}(\mathbf{x})$, find a vector $\bar{\mathbf{x}} = (\bar{x}_1, \ldots, \bar{x}_n)$ such that $p^{(1)}(\bar{\mathbf{x}}) = \ldots = p^{(m)}(\bar{\mathbf{x}}) = 0$.

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- NP hard
- believed to be hard on average even for quantum conputers: suppose we have a probabilistic algorithm A and a subexponential function η, T terminates with an answer to a random instance from $MQ(n, m = an, \mathbb{F}_q)$ in time $\eta(n)$ with probability $\mathrm{negl}(n)$.
- higher order versions (MP for Multivariate Polynomials or PoSSo for Polynomial System Solving) clearly no less hard

However usually no direct reduction to MQ !!

Identification Scheme of Sakumoto et al and MQDSS

An example 5-pass ID scheme depending only on MQ

- $\mathcal{P}$ be a random MQ instance
- Its "polar" form $D\mathcal{P}(\mathbf{x}, \mathbf{y}) := \mathcal{P}(\mathbf{x} + \mathbf{y}) - \mathcal{P}(\mathbf{x}) - \mathcal{P}(\mathbf{y}) - \mathcal{P}(\mathbf{0})$
- $\mathcal{P}(\mathbf{s}) = \mathbf{p}$ is the public key, $\mathbf{s}$ is the secret.
- Peter picks and commits random $(\mathbf{r_0}, \mathbf{t_0}, \mathbf{e_0})$, sets $\mathbf{r_1} = \mathbf{s} - \mathbf{r_0}$ and commits $(\mathbf{r_1}, D\mathcal{P}(\mathbf{t_0}, \mathbf{r_1}) + \mathbf{e_0})$.
- Vera sends random α,
- Peter sets and sends $\mathbf{t_1} := \alpha\mathbf{r_0} - \mathbf{t_0}$, $\mathbf{e_1} := \alpha\mathcal{P}(\mathbf{r_0}) - \mathbf{e_0}$.
- Vera sends challenge Ch, Peter sends $\mathbf{r}_{Ch}$.
- Vera checks the commit of either $(\mathbf{r_0}, \alpha\mathbf{r_0} - \mathbf{t_1}, \alpha\mathcal{P}(\mathbf{r_0}) - \mathbf{e_1})$ or $(\mathbf{r_1}, \alpha(\mathbf{p} - \mathcal{P}(\mathbf{r_1})) - D\mathcal{P}(\mathbf{t_1}, \mathbf{r_1}) - \mathbf{e_1})$.

The Fiat-Shamir transform of this ID scheme is the MQDSS scheme.

Bipolar Construction

- Easily invertible quadratic map $\mathcal{Q} : \mathbb{F}^n \to \mathbb{F}^m$
- Two invertible linear maps $\mathcal{T}(: \mathbb{F}^m \to \mathbb{F}^m)$ and $\mathcal{S}(: \mathbb{F}^n \to \mathbb{F}^n)$
- *Public key*: $\mathcal{P} = \mathcal{T} \circ \mathcal{Q} \circ \mathcal{S}$ supposed to look random
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Encryption Schemes ($m \geq n$)

- TTM-related schemes (all broken)
- PMI+, IPHFE+
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- Simple Matrix (impaired)

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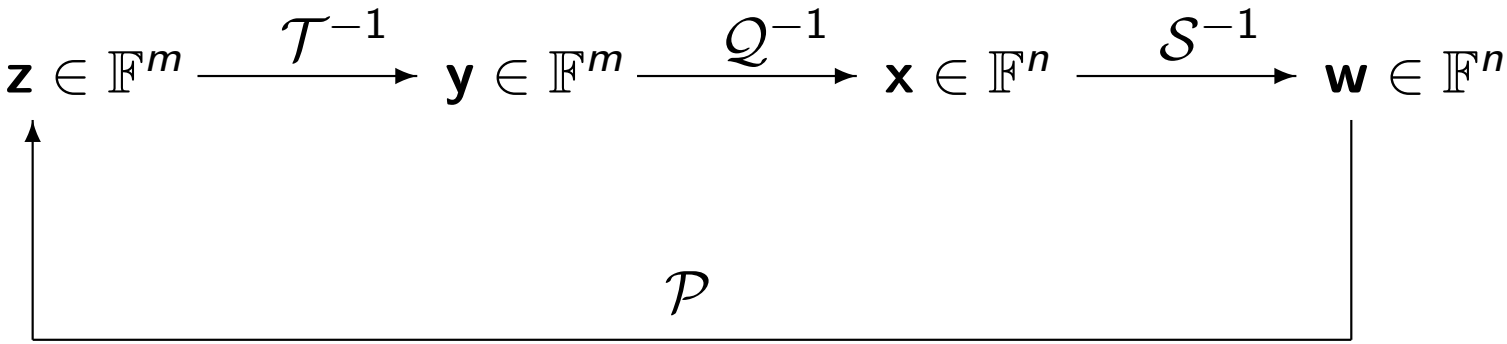
Signature Schemes ($m \leq n$)

- Unbalanced Oil and Vinegar (Rainbow, TTS)
- HFEv- (QUARTZ/Gui)
- pFLASH

Workflow

Decryption / Signature Generation



Encryption / Signature Verification

Isomorphism of Polynomials

Due to the bipolar construction, the security of MPKCs is also based on the

Problem EIP (Extended Isomorphism of Polynomials): Given the public key $\mathcal{P}$ of a multivariate public key cryptosystem, find affine maps $\bar{\mathcal{S}}$ and $\bar{\mathcal{T}}$ as well as quadratic map $\bar{\mathcal{Q}}$ in class $\mathcal{C}$ such that $\mathcal{P} = \bar{\mathcal{T}} \circ \bar{\mathcal{Q}} \circ \bar{\mathcal{S}}$.

$\Rightarrow$ Hardness of the problem depends heavily on the structure of the central map
$\Rightarrow$ In general, not much is known about the complexity
$\Rightarrow$ Security analysis of multivariate schemes is a hard task

Generic (Direct) Attacks

Try to solve the public equation $\mathcal{P}(\mathbf{w}) = \mathbf{z}$ as an instance of the MQ-Problem, all algorithms have exponential running time (for $m \approx n$)

Known Best Generic Algorithms

- For larger q, FXL ("Hybridized XL")
- For $q = 2$, smart enumerative methods

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- For $q = 2$, the Joux-Vitse Algorithm (an XL variant).

Complexity of Direct Attacks

How many equations are needed to meet given levels of security?

security level (bit)	number of equations			
	$\mathbb{F}_2$ *	$\mathbb{F}_{16}$	$\mathbb{F}_{31}$	$\mathbb{F}_{256}$
80	88	30	28	26
100	110	39	36	33
128	140	51	48	43
192	208	80	75	68
256	280	110	103	93

* depending on how we model the Joux-Vitse algorithm

The Problem: Solving Multivariate Systems of Equations

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$\mathcal{MQ}(m, n, q)$ Problem

Find a solution to a system of m quadratic equations
$\ell_1(\mathbf{x}) = \cdots = \ell_m(\mathbf{x}) = 0$ in n variables $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ over $\mathbb{F}_q$.

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- Solving Non-Linear Multivariate systems is conjectured to be exponentially hard in probability. To be precise,

Hardness Assumption of QUAD (Patarin et al)

For any (probabilistic) Turing machine $\mathcal{A}$ trying to solve an MQ system with randomly drawn coefficients where $m/n = c + o(1)$, and sub-exponential function $\eta(n)$, the probability that $\mathcal{A}$ returns the correct answer in time $\eta(n)$ is negligible.

Buchberger Algorithm

As intuitive as High School Algebra

iteratively eliminate the lexicographically leading terms, and a standard elimination to solve two quadratic equations (ℓ_1) and (ℓ_2) in two variables can be considered a simplified case

$$
\begin{array}{lrll}
(\ell_1) = & & x^2 + 2\,yx - y^2 + x - 3\,y - 5 & = 0 \\
(\ell_2) = & & 2\,x^2 + 7\,yx + 3\,y^2 + 4\,x - 7\,y - 11 & = 0 \\
(\ell_3) = & (\ell_2) - 3(\ell_1) = & 3\,yx + 2\,x + 5\,y^2 - y - 1 & = 0 \\
(\ell_4) = & x(\ell_3) - (3y+2)(\ell_1) = & y^2x + 8\,yx + 3\,x - 3\,y^3 - 11\,y^2 - 21\,y - 10 & = 0 \\
(\ell_5) = & (3y+2)(\ell_4) - (y^2+8y+3)(\ell_3) = & 14\,y^4 + 78\,y^3 + 91\,y^2 + 61\,y + 17 & = 0
\end{array}
$$

Buchberger Algorithm

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Pitfalls as pointed out by Lazard 1983

- Degree of equations gets very high.
  Not as much a problem if "overdetermined".
- May be difficult to compute in the right order.
  Advanced rules like "Buchberger's Criteria" to avoid excess pairing.
- Handling multivariate polynomials is intrinsically space-consuming.
  We want to know ahead of time the terms and use matrices.
- Each elimination step actually consists of several S-polynomial steps and the whole can be up to doubly exponential time.
  Avoid duplication of effort by retaining simple multiples.

Introduction to XL

- First suggested by D. Lazard: Gröbner bases, gaussian elimination and resolution of systems of algebraic equations, EUROCAL '83
- Rediscovered by N. Courtois, A. Klimov, J. Patarin, and A. Shamir: Efficient algorithms for solving overdefined systems of multivariate polynomial equations, Eurocrypt 2000
- **eXtends** a polynomial system by multiplying appropriate monomials and **Linearizes** by treating each monomial as an independent variable
  - Turns system solving into linear algebra

$$
\begin{array}{rcl}
xyz + xy + xz + x & = & 0 \\
0 & = & 0 \\
xyz + xz + yz + z & = & 0 \\
xy + x + yz + z & = & 0 \\
xy + xz & = & 0 \\
xyz + xy & = & 0 \\
yz + z & = & 0 \\
xz + x + y + 1 & = & 0 \\
xyz + xy & = & 0 \\
xyz + y & = & 0 \\
xz + z & = & 0 \\
xz + yz + y + z & = & 0
\end{array}
\qquad
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 1 & 0 & 1 & 1 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
xyz \\ xy \\ xz \\ x \\ yz \\ y \\ z \\ 1
\end{bmatrix}
= 0
$$

Figure: Small example of XL solving the system over $\mathbb{F}_2$: $xy + x + yz + z = 0$; $xz + x + y + 1 = 0$; $xz + yz + y + z = 0$. Left column (black): $x, y, z, 1$ times each of the original equations; Right column (blue): Same twelve equations expressed in matrix form; the matrix is a Macaulay matrix.

$$
\begin{aligned}
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Basic XL at Degree D

Let $\mathcal{T}^{(D)} := \{\deg \leq D \text{ monomials}\}$, $T := |\mathcal{T}^{(D)}|$.

- eXtend: first multiply each ℓ_i of degree d_i by every monomial $\mathbf{x}^{\mathbf{b}} := x_1^{b_1} \cdots x_n^{b_n} \in \mathcal{T}^{(D-d_i)}$ to get equations $\mathcal{R}^{(D)}$.
- Linearize: then solve $\mathcal{R}^{(D)}$ as a linear system in all the $\mathbf{x}^{\mathbf{b}} \in \mathcal{T}^{(D)}$. We may be able to solve the system or to reduce down to a univariate equation (say in x_1).

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Criterion for Success

- If the linearized system leads to $1 = 0$ then system is inconsistent.

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Multiply sparse matrix $T \times T$ to vector $2T$ times.

XL Variants

XL2 – simplified F_4

1. **eXtend:** multiply each polynomial $f_1, \ldots, f_m$ by monomials, up to total degree $\leq D$
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FXL – XL with k variables guessed or "hybridized"

large field, with k initial guesses / fixing / "hybridization":

$$\text{Complexity} = \min_k 3q^k \cdot \binom{n-k+d_{\text{XL}}}{d_{\text{XL}}}^2 \cdot \binom{n-k}{d_{XL}}.$$

[generic method with the best asymptotic multiplicative complexity].

XL Variants

XL2 – simplified F_4

1. **eXtend:** multiply each polynomial $f_1, \ldots, f_m$ by monomials, up to total degree $\leq D$
2. **Linearize**: Apply linear algebra to eliminate top level monomials
3. Multiply degree $D-1$ equations by variables, **Eliminate Again**.

FXL – XL with k variables guessed or "hybridized"

XL'

1. **eXtend:** multiply each polynomial $f_1, \ldots, f_m$ by monomials, up to total degree $\leq D$
2. **Linearize**: Apply linear algebra to eliminate all monomials involving the first k variables (and get at least $n-k$ such equations).
3. **Enumerate over** remaining $n-k$ variables.

XL Variants

XL2 – simplified F_4

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2. **Linearize**: Apply linear algebra to eliminate top level monomials
3. Multiply degree $D-1$ equations by variables, **Eliminate Again**.

FXL – XL with k variables guessed or "hybridized"

Joux-Vitse ("Hybridized XL-related method")

1. **eXtend:** multiply each polynomial $f_1, \ldots, f_m$ by monomials, up to total degree $\leq D$
2. **Linearize**: Apply linear algebra to eliminate all monomials of total degree ≥ 2 in the first k variables (and get at least k such equations).
3. **Fix** $n-k$ variables, solve for the initial k in linear equations.

More Advanced Gröbner Bases Algorithms

- find a "nice" basis of the ideal $\langle f_1, \ldots, f_m \rangle$
- first studied by B. Buchberger
- later improved by Faugére et al. (F_4, F_5)
- With linear algebra constant $2 < \omega \leq 3$.

$$\text{Complexity}(q, m, n) = O\left(\binom{n + d_{\text{reg}} - 1}{d_{\text{reg}}}^{\omega}\right) \qquad \text{(for larger } q\text{)}$$

- "Hybridized":

$$\text{Complexity}(q, m, n) = \min_k \; q^k \cdot O\left(\binom{n - k + d_{\text{reg}} - 1}{d_{\text{reg}}}^{\omega}\right)$$

Do not blithely set $\omega = 2$ here

Even if $\omega \to 2$, there is a huge constant factor which cannot be neglected.

Facts of Life for XL on generic systems

Denote by $[u]s$ the coefficient of the monomial u in the series expansion s.

Characteristic 0 case

Suppose having considered k equations $(\ell_1 = \ell_2 = \cdots = \ell_k = 0)$ there are s_D monomials at degree D or lower which are linearly independent. Consider another equation $\ell = \ell_{k+1} = 0$ of degree $d = d_{k+1}$ to the system. Clearly for each monomial u of degree $D - d$ or lower, $u\ell = 0$ makes another relation among the monomials of degree D or lower.

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So assuming "obvious" regularity conditions on the (ℓ_i)'s, there are $s'_D = s_D - s_{D-d}$ remaining degrees of linear independence.
I.e., if $\widehat{s'}(x) := \sum_i s'_i x^i$ and $\widehat{s}(x) := \sum_i s_i x^i$, then $\widehat{s'}(x) = \widehat{s}(x) \cdot (1 - x^d)$

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Characteristic q case where $\ell_i \cdot (\ell_i^{q-1} - 1) = 0$

If everything is in $\mathbb{F}_q$, we would get $\widehat{s'}(x) = \widehat{s}(x) \cdot (1 - x^d)/(1 - x^{dq})$

Facts of Life for XL on generic systems (cont.)

Assuming "the usual" regularity conditions on the (ℓ_i), and $\deg \ell_i := d_i$,

$$\#\text{ monomials: } T = [t^D]\left((1-t^q)^n(1-t)^{-(n+1)}\right); \tag{1}$$

$$\#\text{ free monoms: } T - I \geq [t^D]\left(\frac{(1-t^q)^n}{(1-t)^{n+1}}\prod_{i=1}^{m}\left(\frac{1-t^{d_i}}{1-t^{qd_i}}\right)\right). \tag{2}$$

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is the reduced case for large fields ($q > D$).

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$C_{XL} \approx 3kT^2(c_0 + c_1 \lg T)$ memory or arithmetic operations, using modified Wiedemann algorithms (k is average number of terms per equation).

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T is singly exponential if $D \propto n$. In general people write $C = T^\omega$ here, ω is the linear algebra constant.

$\mathbf{F_4}/\mathbf{F_5}$/XL2 vs XL for generic systems

Small Fields ($T = [t^D]\ \left((1-t^q)^n(1-t)^{-(n+1)}\right)$;)

$$D_{XL} = \min\left\{D : [t^D]\ \left(\frac{(1-t^q)^n}{(1-t)^{n+1}}\ \prod_{i=1}^{m}\left(\frac{1-t^{d_i}}{1-t^{qd_i}}\right)\right) \leq 0\right\}.$$

$$D_{reg} = \min\left\{D : [t^D]\ \left(\frac{(1-t^q)^n}{(1-t)^{n}}\ \prod_{i=1}^{m}\left(\frac{1-t^{d_i}}{1-t^{qd_i}}\right)\right) < 0\right\}.$$

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Large Fields ($q > D$, $T = \binom{n+D}{D}$)

$$D_{XL} = \min\left\{D : [t^D]\ \left((1-t)^{m-n-1}\ (1+t)^m\right) \leq 0\right\},$$

$$D_{reg} = \min\left\{D : [t^D]\ \left((1-t)^{m-n}\ (1+t)^m\right) < 0\right\}.$$

$\mathbf{F_4}/\mathbf{F_5}$/XL2 vs XL for generic systems

Small Fields ($T = [t^D]\ \left((1-t^q)^n(1-t)^{-(n+1)}\right)$;)

$$D_{XL} = \min\left\{D : [t^D]\ \left(\frac{(1-t^q)^n}{(1-t)^{n+1}}\ \prod_{i=1}^{m}\left(\frac{1-t^{d_i}}{1-t^{qd_i}}\right)\right) \leq 0\right\}.$$

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$$D_{reg} = \min\left\{D : [t^D]\ \left((1-t)^{m-n}\ (1+t)^m\right) < 0\right\}.$$

$\mathbb{F}_2$ case ($T = \sum_{i=0}^{D}\binom{n}{i}$)

$$D_{XL} = \min\left\{D : [t^D]\ \left((1-t)^{-1}\ (1+t)^n\ (1+t^2)^{-m}\right) \leq 0\right\},$$

$$D_{reg} = \min\left\{D : [t^D]\ \left(\ (1+t)^n\ (1+t^2)^{-m}\right) < 0\right\}.$$

Remarks

Every cryptosystem can be represented as a set of nonlinear multivariate equations

- Direct attacks can be used in the cryptanalysis of other cryptographic schemes (in particular block and stream ciphers)
- The MQ (or PoSSo) Problem can be seen as one of the central problems in cryptography

Post-Quantum-ness of MQ

MQ is quantum-resistant: the best Grover-based quantum attack against n-bits of input takes $2^{\frac{n}{2}+1}n^3$ time.

Features of Multivariate Cryptosystems

Advantages

- resistant against attacks with quantum computers
- very fast (much faster than RSA)
- only simple arithmetic operations required
  $\Rightarrow$ can be implemented on low cost devices
  $\Rightarrow$ suitable for security solutions for the IoT
- many practical signature schemes (UOV, Rainbow, HFEv-, ...)
- short signatures (e.g. 120 bit signatures for 80 bit security)

Disadvantages

- large key sizes (public key size $\sim 10 - 100$ kB)
- no security proofs
- mainly restricted to digital signatures

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Multivariate Quadratic Public-Key Cryptography Part 2: Big Field Schemes

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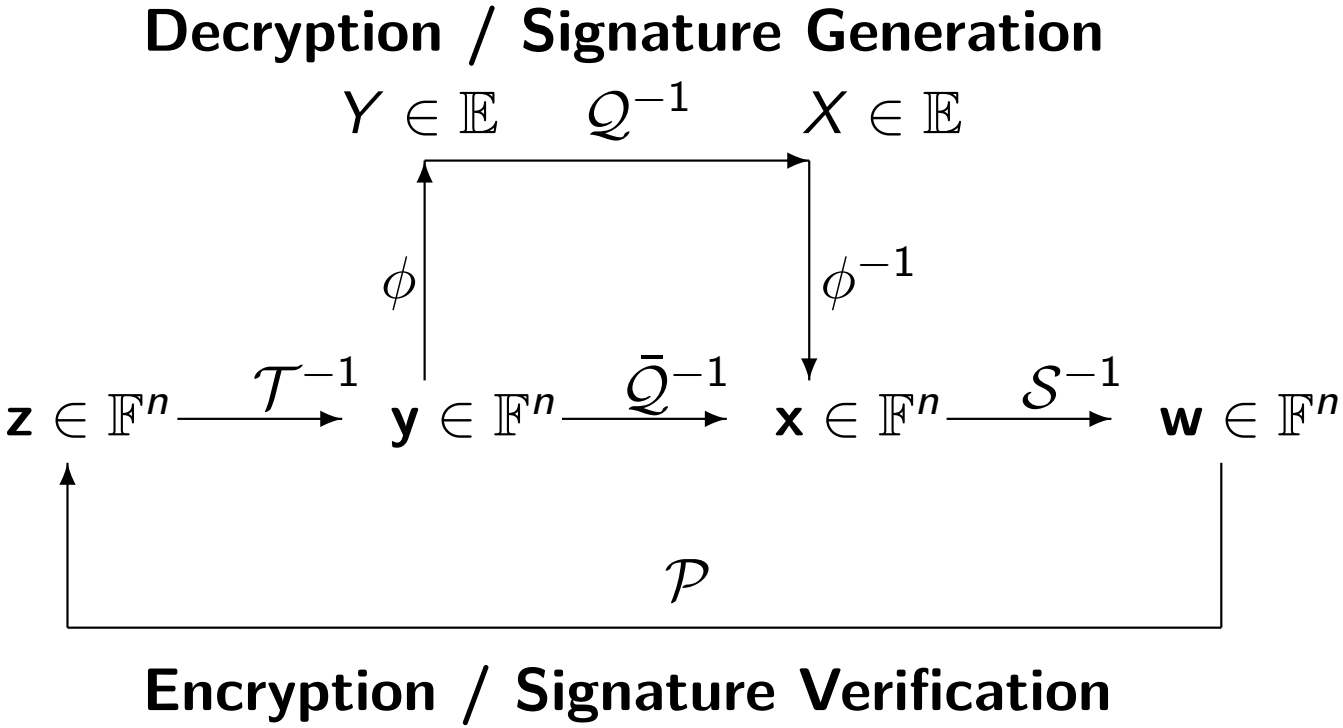
Taipei, Taiwan
Wednesday, 27.06.2018

Big Field Schemes



Extension Fields

- $\mathbb{F}_q$: finite field with q elements
- $g(X)$ irreducible polynomial in $\mathbb{F}[X]$ of degree n
  $\Rightarrow \mathbb{F}_{q^n} \cong \mathbb{F}[X]/\langle g(X) \rangle$ finite field with q^n elements
- isomorphism $\phi : \mathbb{F}_q^n \to \mathbb{F}_{q^n}$, $(a_1, \ldots, a_n) \mapsto \sum_{i=1}^{n} a_i \cdot X^{i-1}$
- Addition in $\mathbb{F}_{q^n}$: Addition in $\mathbb{F}_q[X]$
- Multiplication in $\mathbb{F}_{q^n}$: Multiplication in $\mathbb{F}_q[X]$ modulo $g(X)$

The Matsumoto-Imai Cryptosystem (1988) [MI88]

- $\mathbb{F}_q$: finite field of characteristic 2
- degree n extension field $\mathbb{E} = \mathbb{F}_{q^n}$
- isomorphism $\phi : \mathbb{F}_q^n \to \mathbb{E}$
- MI parameter $\theta \in \mathbb{N}$ with

$$\gcd(q^\theta + 1, q^n - 1) = 1.$$

Key Generation

- *central map* $\mathcal{Q} : \mathbb{E} \to \mathbb{E}$, $X \mapsto X^{q^\theta+1} \Rightarrow \mathcal{Q}$ is bijective
- choose 2 invertible linear or affine maps $\mathcal{S}, \mathcal{T} : \mathbb{F}^n \to \mathbb{F}^n$
- *public key*: $\mathcal{P} = \mathcal{T} \circ \phi^{-1} \circ \mathcal{Q} \circ \phi \circ \mathcal{S} : \mathbb{F}^n \to \mathbb{F}^n$ quadratic multivariate map
- use the extended Euclidian algorithm to compute $h \in \mathbb{N}$ with

$$h \cdot \theta \equiv 1 \bmod q^n - 1$$

- *private key*: $\mathcal{S}, \mathcal{T}$

Both Encryption and Signature

Encryption or Verification

Given: plaintext or signature $\mathbf{w} \in \mathbb{F}^n$ or Compute $\mathbf{z} \in \mathbb{F}^n$ by $\mathbf{z} = \mathcal{P}(\mathbf{w})$.
This is the ciphertext. Or the result to be matched against a hash digest.

Decryption or Signing

Given: ciphertext or hash digest $\mathbf{z} \in \mathbb{F}^n$

1. Compute $\mathbf{y} = \mathcal{T}^{-1}(\mathbf{z})$.
2. Compute $Y = \phi(\mathbf{y}) \in \mathbb{E}$
3. Compute $X = \mathcal{Q}^{-1}(Y)$ by $X = Y^h$
4. Compute $\mathbf{x} = \phi^{-1}(X) \in \mathbb{F}^n$
5. Compute the plaintext or signature $\mathbf{w} \in \mathbb{F}^n$ by $\mathbf{w} = \mathcal{S}^{-1}(\mathbf{x})$.

Linearization, a Message Recovery Attack [Pa95]

Given public key $\mathcal{P}$, $\mathbf{z}^\star \in \mathbb{F}^n$, find plaintext $\mathbf{w}^\star \in \mathbb{F}^n$, s.t. $\mathcal{P}(\mathbf{w}^\star) = \mathbf{z}^\star$

Proposed by J. Patarin in 1995

Taking the $q^\theta - 1$ st power of $Y = X^{q^\theta+1}$ and multiplying with XY yields

$$X \cdot Y^{q^\theta} = X^{q^{2\theta}} \cdot Y$$

$\Rightarrow$ bilinear equation in X and Y, hence, same in $\mathbf{w}$ and $\mathbf{z}$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij} w_i z_j + \sum_{i=1}^{n} \beta_i w_i + \sum_{j=1}^{n} \gamma_j z_j + \delta = 0. \quad (\star)$$

1. Compute $N \geq \frac{(n+1)\cdot(n+2)}{2}$ pairs $(\mathbf{z}^{(k)}/\mathbf{w}^{(k)})$ and substitute into $(\star)$.
2. Solve the resulting linear system for the coefficients α_{ij}, β_i, γ_j and δ.
   $\Rightarrow$ n bilinear equations in $w_1, \ldots, w_n, z_1, \ldots, z_n$
3. Substitute $\mathbf{z}^\star$ into these bilinear equations and solve for $\mathbf{w}^\star$.

C^{*-} Schemes

C^{*-} schemes are C^* schemes with a truncated public key [PGC98]

Construction of a C^{*-} scheme

(n, θ, r) are the parameters of the scheme

1. Generate a C^* with parameters (n, θ): $\mathcal{Q}(x) = x^{1+q^\theta}$
2. Remove the last r polynomials from the public key

$$T \circ \mathcal{Q} \circ S = \begin{cases} p_1(x_1, \ldots, x_n) \\ \vdots \\ \vdots \\ p_n(x_1, \ldots, x_n) \end{cases} \xrightarrow{\Pi} \begin{cases} p_1(x_1, \ldots, x_n) \\ \vdots \\ p_{n-r}(x_1, .., x_n) \end{cases} = \Pi \circ \mathcal{P}$$

SFLASH$=C^{*-}$ $(\mathbb{F}_{128},37,26)$

Signing

1. Append r random values μ to the message m to be signed
2. Find a preimage σ of (m,μ) by $T \circ \mathcal{Q} \circ S$ using S, T
3. Such a preimage always exists since a C^* monomial is bijective
4. σ is a valid signature since $\Pi \circ \mathcal{P}(\sigma) = m$

Parameters (n,θ) must define a bijective C^*

$$\mathcal{Q}(x) = x^{1+q^\theta}$$

- $\mathcal{Q}$ is bijective when $\gcd(q^\theta+1, q^n-1) = 1 \quad (q = 2^k)$
- This condition is equivalent to n/d odd where $d = \gcd(n,\theta)$

Parameter r should not be too small; more precisely...

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Parameter r should not be too small; more precisely...

$q^r \geq 2^b$ to avoid a possible recomposing attack

Skew-Symmetry: C^{*-} Attack by Dubois (2007)

First attack requires $d = \gcd(n, \theta) > 1$, but isn't necessary

Take any $\zeta \in (\mathbb{F}_{q^n})^*$, then $D\mathcal{Q}(\zeta a, x) + D\mathcal{Q}(a, \zeta x) = L(\zeta) D\mathcal{Q}(a, x)$, implying that if $\mathrm{M}_\zeta = S^{-1} \circ M_\zeta \circ S$, where M_ζ means multiplying by ζ, then if H_i are symmetric matrices of the public key polynomials p_i, we should have

$$\mathrm{span}\{\mathrm{M}_\zeta^T H_i + H_i \mathrm{M}_\zeta : i = 1 \cdots n\} = \mathrm{span}\{H_i : i = 1 \cdots n\}.$$

Heuristic Argument by Shamir et al

pick three random linear combinations $\sum_{i=1}^{n-r} b_i (\mathrm{M}_\zeta^T H_i + H_i \mathrm{M}_\zeta)$ and demand that they fall in $S = \mathrm{span}\{H_i : i = 1 \cdots n - r\}$, then

1. there is a good chance to find a nontrivial M_ζ
2. this matrix really correspond to a multiplication by ζ in $\mathbb{F}_{q^n}$;
3. the skew-symmetric action of this M_ζ on the H_i leads to matrices in $\mathrm{span}\{H_i : i = 1 \cdots n\} \backslash S$.

Net Result of Differential Attacks

End of SFLASH

The heuristic argument holds under comprehensive tests and SFLASH and in fact all C^{*-} are comprehensively broken!! Later a slightly more complex but very similar argument was used to break the similar ℓIR signature scheme (PKC 2008) by Fouque et al.

A Defense In One Sentence

When we restrict to a subspace H of $\mathbb{F}_{q^n}$, the only maps that satisfy the symmetry properties (required of the differential attacks) happens to be the same ones in $\mathbb{F}_{q^n}$ that leaves H invariant.

Projections block differential attacks

All symmetry disappears from hyperplane-restricted C^{*-}'s. Differential Attacks verified not to work. This is further studied by Smith et al.

Prefixed C^{*-} signature scheme

Natural restriction of $\mathcal{Q}$ to hyperplane = set coordinate to 0

Start from a C^* scheme with $\mathcal{Q}(x) = x^{1+q^\theta}$ with secret linear maps S and T. Let r and s be two integers between 0 and n. Let T^- be the projection of T on the last r coordinates and S^- be the restriction of S to the first $n-s$ coordinates. $\mathcal{P} = T^- \circ \mathcal{Q} \circ S^-$ is the public key and S^{-1} and T^{-1} are used as the secret key.

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Inversion

To find $\mathcal{P}^{-1}(m)$ for $m \in \mathbb{F}_q^{n-r}$, the legitimate user first pads m with a random vector m' of $(\mathbb{F})^r$ and compute the preimage of (m, m') by $T^{-1} \circ \mathcal{Q}^{-1} \circ S^{-1}$. If this element has its last s coordinates to 0, then its $n-s$ first coordinates are a valid signature for m. Otherwise, he discards this element and tries with another m'. When $r > s$, the process ends with probability 1 and costs on average q^s inversions of $\mathcal{Q}$.

pFLASH (C^{*-} p, prefixed C^{*-})

Choosing Parameters

n, θ, r are chosen following the rationales for C^{*-} schemes. As signing is q^s times slower, we prefer $s = 1$ and q small. However, if q is chosen small, at constant blocksize this requires a larger value of n and therefore larger keys.

Realistic 80-bit Parameters: pFLASH($\mathbb{F}_{16}$,62-1,40)

As a possible trade-off, the original proposers suggested pFLASH with $q = 2^4$, $n = 74$, $\theta = 11$, $r = 22$ and $s = 1$ (we call this pFLASH($\mathbb{F}_{16}$,74-1,56)). It has (as expected) a bigger secret key of 5.4kB and signs in line with expectations of $\sim 16\times$ time of SFLASH. Currently Smith et al suggests pFLASH($\mathbb{F}_{16}$,62-1,40).
One big plus for $q = 2^4$ is to compute over $\mathbb{F}_{2^8}$ until the last step.

Larger pFLASH Parameters at 128 and 256 bits

We suggest pFLASH($\mathbb{F}_{16}$,96-1,64) and pFLASH($\mathbb{F}_{16}$,192-1,128).

The HFE Cryptosystem [Pa96]

- " Hidden Field Equations", proposed by Patarin in 1995
- BigField Scheme, can be used both for encryption and signatures
- finite field $\mathbb{F}$, extension field $\mathbb{E}$ of degree n, isomorphism $\phi : \mathbb{F}^n \to \mathbb{E}$

Original HFE

- central map $\mathcal{Q} : \mathbb{E} \to \mathbb{E}$ (not bijective, invert using Berlekamp Algorithm).

$$\mathcal{Q}(X) = \sum_{0 \le i \le j}^{q^i + q^j \le D} \alpha_{ij} X^{q^i + q^j} + \sum_{i=0}^{q^i \le D} \beta_i \cdot X^{q^i} + \gamma$$

  $\Rightarrow \bar{\mathcal{Q}} = \phi^{-1} \circ \mathcal{Q} \circ \phi : \mathbb{F}^n \to \mathbb{F}^n$ quadratic

- degree bound D needed for efficient decryption / signature generation
- linear maps $\mathcal{S}, \mathcal{T} : \mathbb{F}^n \to \mathbb{F}^n$
- *public key*: $\mathcal{P} = \mathcal{T} \circ \bar{\mathcal{Q}} \circ \mathcal{S} : \mathbb{F}^n \to \mathbb{F}^n$
- *private key*: $\mathcal{S}$, $\mathcal{Q}$, $\mathcal{T}$

Decryption and Signature Generation

Signing message d

1. Use hash function $\mathcal{H} : \{0,1\}^\star \to \mathbb{F}^n$ to compute $\mathbf{z} = \mathcal{H}(d)$
2. Compute $\mathbf{y} = \mathcal{T}^{-1}(\mathbf{z}) \in \mathbb{F}^n$ and $Y = \phi(\mathbf{y}) \in \mathbb{E}$
3. Solve $\mathcal{Q}(X) = Y$ over $\mathbb{E}$ via Berlekamp's algorithm
4. Compute $\mathbf{x} = \phi^{-1}(X) \in \mathbb{F}^n$ and $\mathbf{w} = \mathcal{S}^{-1}(\mathbf{x})$

Signature: $\mathbf{w} \in \mathbb{F}^n$.

Decryption proceeds similarly, but ...

- Signature generation process does not output a signature for every input message $\Rightarrow$ need to append a counter to the message d
- Decryption is not unique $\Rightarrow$ need disambiguation in the plaintext.

MinRank Attack against HFE

Look in extemsion field $\mathbb{E}$ (Kipnis and Shamir [KS99])

- the linear maps $\mathcal{S}$ and $\mathcal{T}$ relate to univariate maps $\mathcal{S}^\star(X) = \sum_{i=1}^{n-1} s_i \cdot X^{q^i}$ amd $\mathcal{T}^\star(X) = \sum_{i=1}^{n-1} t_i \cdot X^{q^i}$, with $s_i,\ t_i \in \mathbb{E}$.
- the public key $\mathcal{P}^\star$ can be expressed as $\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} p_{ij}^\star X^{q^i+q^j} = \underline{X} \cdot P^\star \cdot \underline{X}^T$,
- Components of $P^\star$ can be found by polynomial interpolation.
- Solve MinRank problem over $\mathbb{E}$.

No need to look in $\mathbb{E}$ (Bettale et al)

Perform the MinRank attack without recovering $\mathcal{P}^\star \Rightarrow$ HFE can be broken by using a MinRank problem over the base field $\mathbb{F}$.

$$\text{Complexity}_{\text{MinRank}} = \binom{n+r}{r}^\omega$$

with $2 < \omega \leq 3$ and $r = \lfloor \log_q(D-1) \rfloor + 1$.

Direct Attacks

- J-C Faugère solved HFE Challenge 1 (HFE over GF2, $d = 96$) in 2002
- Empirically HFE systems can be solved much faster than random
- Ding-Hodges Upper bound for d_{reg}

$$d_{reg} \leq \begin{cases} \frac{(q-1)\cdot(r-1)}{2} + 2 & q \text{ even and } r \text{ odd}, \\ \frac{(q-1)\cdot r}{2} + 2 & \text{otherwise.} \end{cases},$$

with $r = \lfloor \log_q(D-1) \rfloor + 1$.

$\Rightarrow$ Basic version of HFE is not secure

Variant Schemes

- Encryption Schemes IPHFE+ (inefficient), ZHFE (broken).
- Signature Schemes HFEv- (QUARTZ/GUI), MHFEv- (Broken)

HFEv-

- finite field $\mathbb{F}$, extension field $\mathbb{E}$ of degree n, isomorphism $\phi : \mathbb{F}^n \to \mathbb{E}$
- central map $\mathcal{Q} : \mathbb{F}^v \times \mathbb{E} \to \mathbb{E}$, where the β_i and γ are affine.

$$\mathcal{Q}(X) = \sum_{0 \le i \le j}^{q^i + q^j \le D} \alpha_{ij} X^{q^i + q^j} + \sum_{i=0}^{q^i \le D} \beta_i(v_1, \ldots, v_v) \cdot X^{q^i} + \gamma(v_1, \ldots, v_v)$$

$\Rightarrow \bar{\mathcal{Q}} = \phi^{-1} \circ \mathcal{Q} \circ (\phi \times \mathrm{id}_v)$ quadratic map: $\mathbb{F}^{n+v} \to \mathbb{F}^n$

- linear maps $\mathcal{T} : \mathbb{F}^n \to \mathbb{F}^{n-a}$ and $\mathcal{S} : \mathbb{F}^{n+v} \to \mathbb{F}^{n+v}$ of maximal rank
- *public key*: $\mathcal{P} = \mathcal{T} \circ \bar{\mathcal{Q}} \circ \mathcal{S} : \mathbb{F}^{n+v} \to \mathbb{F}^{n-a}$
- *private key*: $\mathcal{S}$, $\mathcal{Q}$, $\mathcal{T}$

Signing Message digest **z**

1. Compute $\mathbf{y} = \mathcal{T}^{-1}(\mathbf{z}) \in \mathbb{F}^n$ and $Y = \phi(\mathbf{y}) \in \mathbb{E}$
2. Choose random values for the vinegar variables $v_1, \ldots, v_v$
   Solve $\mathcal{Q}_{v_1,\ldots,v_v}(X) = Y$ over $\mathbb{E}$ via Berlekamps algorithm.
3. Compute $\mathbf{x} = \phi^{-1}(X) \in \mathbb{F}^n$ and signature $\mathbf{w} = \mathcal{S}^{-1}(\mathbf{x} || v_1 || \ldots || v_v)$.

Security vs. Efficiency

Main Attacks

- MinRank Attack $\mathrm{Rank}(F) = r + a + v$

  $$\Rightarrow \mathrm{Compl}_{\mathrm{MinRank}} = \binom{n + r + a + v}{r + a + v}^{\omega}$$

- Direct attack [DY13]

$$d_{reg} \leq \begin{cases} \frac{(q-1)\cdot(r+a+v-1)}{2} + 2 & q \text{ even and } r + a \text{ odd}, \\ \frac{(q-1)\cdot(r+a+v)}{2} + 2 & \text{otherwise.} \end{cases},$$

with $r = \lfloor \log_q(D-1) \rfloor + 1$ and $2 < \omega \leq 3$.

Efficiency

Rate determining step: solving X from a univariate equation of degree D.

$$\mathrm{Complexity}_{\mathrm{Berlekamp}} = \mathcal{O}(D^3 + n \cdot D^2)$$

How to define a HFEv- like scheme over $\mathbb{F}_2$ [PCY+15]?

Collision Resistance of the hash function

To cover a hash value of k bit, the public key of a pure HFEv- scheme has to contain at least k equations over $\mathbb{F}_2$. $\Rightarrow$ public key $> k^3/2$ bits

security level	80	100	128	192	256
# equations	100	200	256	384	512
pubkey size (kB)	>250	> 500	> 1000	> 3000	> 8000

QUARTZ

- standardized by Courtois, Patarin in 2002
- HFEv$^-$ with $\mathbb{F} = \mathrm{GF}(2)$, $n = 103$, $D = 129$, $a = 3$ and $v = 4$
- public key: quadratic map $\mathcal{P} = \mathcal{T} \circ \mathcal{Q} \circ \mathcal{S} : \mathrm{GF}(2)^{107} \to \mathrm{GF}(2)^{100}$
- Prevent birthday attacks $\Rightarrow$ Generate four HFEv$^-$ signatures (for $\mathbf{w}$, $\mathcal{H}(\mathbf{w}|00)$, $\mathcal{H}(\mathbf{w}|01)$ and $\mathcal{H}(\mathbf{w}|11)$)
- Combine them to a single signature of length $(n-a) + 4 \cdot (a+v) = 128$ bit

GUI (Generalization of QUARTZ) Signature Generation

Input: HFEv- private key $(\mathcal{S}, \mathcal{Q}, \mathcal{T})$ message $\mathbf{d}$, repetition factor k
Output: signature $\sigma \in \mathbb{F}_2^{(n-a)+k(a+v)}$

1: $\mathbf{h} \leftarrow$ SHA-256($\mathbf{d}$)
2: $S_0 \leftarrow \mathbf{0} \in \mathrm{GF}(2)^{n-a}$
3: **for** $i = 1$ to k **do**
4: $\quad D_i \leftarrow$ first $n - a$ bits of $\mathbf{h}$
5: $\quad (S_i, X_i) \leftarrow \mathrm{HFEv}-^{-1}(D_i \oplus S_{i-1})$
6: $\quad \mathbf{h} \leftarrow$ SHA-256($\mathbf{h}$)
7: **end for**
8: $\sigma \leftarrow (S_k||X_k|| \ldots ||X_1)$
9: **return** σ

Note that if the equation has zero or more than 2 equations, then we discard those vinegars and try again.

Signature Verification

Input: HFEv- public key $\mathcal{P}$, message **d**, repetition factor k, signature $\sigma \in \mathbb{F}_2^{(n-a)+k(a+v)}$

Output: **TRUE** or **FALSE**

1: $\mathbf{h} \leftarrow$ SHA-256(**d**)
2: $(S_k, X_k, \ldots, X_1) \leftarrow \sigma$
3: **for** $i = 1$ to k **do**
4: $\quad D_i \leftarrow$ first $n - a$ bits of **h**
5: $\quad \mathbf{h} \leftarrow$ SHA-256(**h**)
6: **end for**
7: **for** $i = k - 1$ to 0 **do**
8: $\quad S_i \leftarrow \mathcal{P}(S_{i+1}||X_{i+1}) \oplus D_{i+1}$
9: **end for**
10: **if** $S_0 = \mathbf{0}$ **then**
11: $\quad$ **return TRUE**
12: **else**
13: $\quad$ **return FALSE**
14: **end if**

Parameters for HFEv- (GUI) over $\mathbb{F}_2$?

Parameters are set by the complexity of MinRank and direct attacks

- For the complexity of the MinRank attack we have a concrete formula
- For the direct attack, we only have an upper bound on d_{reg}.

$$d_{reg} \leq \begin{cases} \frac{(q-1)\cdot(r+a+v-1)}{2} + 2 & q \text{ even and } r+a \text{ odd}, \\ \frac{(q-1)\cdot(r+a+v)}{2} + 2 & \text{otherwise.} \end{cases} \quad (\star)$$

Experiments show that these estimate for d_{reg} is reasonably tight.

Parameter Choice of HFEv- over $\mathbb{F}_2$

Efficiency $\Rightarrow$ Choose D as small as possible

- $D = 5 \Rightarrow r = \lfloor \log_2(D-1) \rfloor + 1 = 3$
- $D = 9 \Rightarrow r = \lfloor \log_2(D-1) \rfloor + 1 = 4$
- $D = 17 \Rightarrow r = \lfloor \log_2(D-1) \rfloor + 1 = 5$

Increase a and v to reach the required security level

Choose a and v as equal as possible, i.e. $0 \leq v - a \leq 1$.

Quantum Attacks and Impact

A determined multivariate system of m equations over $\mathbb{F}_2$ can be solved using $2^{m/2} \cdot 2 \cdot m^3$ operations using a quantum computer.

- This does not affect signatures in general because the hashes are typically twice as wide as the design security.
- Alas, this wipes out some of GUI's gains.

$\Rightarrow$ very large public key size

quantum security level	80	100	128	192	256
min # equations	117	155	208	332	457

Current Recommended Quantum-Safe Parameters

Cl. Security level (bit)	Q. Security level (bit)		public key size (kB)	private key size (kB)	signature size (bit)
128	108	Gui ($\mathbb{F}_2$,184,33,16,16,3)	416.3	19.1	392
192	170	Gui ($\mathbb{F}_2$,312,129,24,20,3)	1995.1	59.3	548
256	236	Gui ($\mathbb{F}_2$,448,513,32,28,2)	5789.2	155.9	724

HFEv- - Summary

- very short (pre-quantum) signatures
- security well respected
- conflict between security and efficiency
- restricted to very small fields, hence very large keys

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