Algebraic approaches for the Elliptic Curve Discrete Logarithm Problem over Prime Fields

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Elliptic Curve Discrete Logarithm Problem

- **Elliptic Curve Discrete Logarithm Problem** (ECDLP)
  Let K a finite field and let E be an elliptic curve over K.
  Let $P \in E(K)$ and let $Q \in G :=< P >$.
  Find $k \in \mathbb{Z}$ such that $Q = kP$.
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- In practice K is a prime field, a binary field with prime degree extension, or $\mathbb{F}_{p^n}$ with n relatively small
- **Elliptic Curve Cryptography secure** $\Rightarrow$ **ECDLP hard**

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- Index calculus approaches being developed since 2004, but mostly focused on extension fields
- Our goal : extend previous index calculus algorithms to ECDLP over prime fields

Outline

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New variants for curves over prime fields

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Index Calculus for Elliptic Curves

1. Fix $m \in \mathbb{Z}$, and fix $V \subset K$ with $|V|^m \approx K$
   Define a *factor basis*

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2. Compute about $|\mathcal{F}|$ *relations*

$$a_i P + b_i Q \;=\; P_{i,1} + P_{i,2} + \ldots + P_{i,m}$$

   with $P_{i,j} \in \mathcal{F}$

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3. Linear algebra on relations gives $aP + bQ = 0$

Relation search : Semaev polynomials

- **Semaev polynomials** relate the x-coordinates of points that sum up to 0 :
  $S_r(x_1, \ldots, x_r) = 0$
  $\Leftrightarrow\ \exists (x_i, y_i) \in E(\bar{K})$ s.t. $(x_1, y_1) + \cdots + (x_r, y_r) = 0$

- **Relation search**
  - Compute $(X, Y) := aP + bQ$ for random a, b
  - Search for $x_i \in V$ with $S_{m+1}(x_1, \ldots, x_m, X) = 0$
  - For any such solution, find corresponding y_i values

Existing Variants

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  - Reduction to polynomial system over $\mathbb{F}_q$
  - Generic bounds give $L_{q^n}(2/3)$ complexity if $q = L_{q^n}(2/3)$

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- Diem, FPPR, P-Quisquater
  - $K = \mathbb{F}_{2^n}$ and V a vector space of K over $\mathbb{F}_2$
  - Reduction to polynomial system over $\mathbb{F}_2$
  - Experiments suggest system "somewhat easy"

Relation search : Weil Descent

- For each relation solve a *generalized root-finding problem*

  Given $f \in \mathbb{F}_{q^n}[x_1, \ldots, x_m]$ *and vector space* $V \subset \mathbb{F}_{q^n}$*,*
  find $x_i \in V$ *such that* $f(x_1, \ldots, x_m) = 0$

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- Solved by Weil Descent : reduction to polynomial system
  - Fix a basis for V over $\mathbb{F}_q$
  - Introduce variables $x_{ij} \in \mathbb{F}_q$ with $x_i = \sum_j x_{ij} v_j$
  - See single equation $f\left(\sum_j x_{1j} v_j, \ldots, \sum_j x_{mj} v_j\right) = 0$
    over $\mathbb{F}_{q^n}$ as a system of n equations over $\mathbb{F}_q$

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- In binary case asymptotic complexity is not clear, and practical complexity is poor
- Not clear how to extend to prime fields : no subspace available and we a priori want small degree equations

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Previous index calculus algorithms for ECDLP

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Main idea

- Find low degree rational maps L_j such that

$$\#\{x \in \mathbb{F}_p \mid L(x) = L_{n'} \circ \ldots \circ L_1(x) = 0\} \approx \prod \deg L_j \approx p^{1/m}$$

- Define $V = \{x \in \mathbb{F}_p \mid L(x) = 0\}$
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- Relation search : solve the polynomial system

$$\begin{cases} S_{m+1}(x_{11}, \ldots, x_{m1}, X) = 0 & \\ x_{i,j+1} = L_j(x_{i,j}) & i = 1, \ldots, m; j = 1, \ldots, n' - 1 \\ 0 = L_{n'}(x_{i,n'}) & i = 1, \ldots, m. \end{cases}$$

Remarks

- One can write similar systems in binary cases, and show they are equivalent to Weil descent systems
- *Precomputation* of the maps L_j can a priori be used for any DLP defined over any curve over the same field

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- *Precomputation* of the maps L_j can a priori be used for any DLP defined over any curve over the same field
- Remaining of the talk :
  - How to compute the maps L_j ?
  - How to solve the system ?

Finding good maps : $p-1$ "smooth"

- Suppose $\mathbf{p-1} = S \cdot N'$ with $S \approx p^{1/m}$ **smooth**
- We want low degree rational maps L_j such that

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- Take $L(X) = X^S - 1$ and V subgroup of order S in $\mathbb{F}_p^*$
- If $S = \prod_{j=1}^{n'} q_j$ take $L_j(X) = X^{q_j}$ and $L_{n'}(X) = X^{q_{n'}} - 1$

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- Remark : NIST P-224 curve satisfies

$$p-1=2^{96}\cdot N'$$

Finding good maps : isogeny Kernels

- Find an **auxiliary curve** E' with $\#E'(\mathbb{F}_p) = S \cdot N'$ and $S = \prod_{j=1}^{n'} q_j \approx p^{1/m}$ smooth
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- Take L_j the x-coordinate part of φ_j, except for $L_{n'}$ taken in a slightly different way

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- Method 1 needs at most $\approx |\mathcal{F}|$ trials on average
- Method 2 more efficient when you can chose p yourself (kind of trapdoor)

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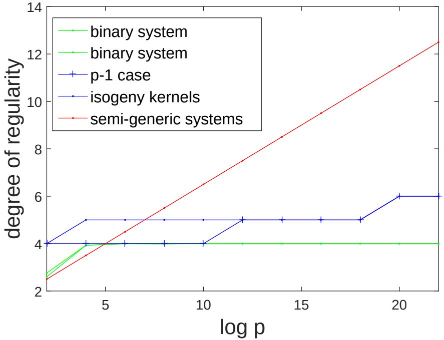
- Low degree equations, block triangular structure
- mn' variables and $mn' + 1$ equations
- Seems reasonable to expect dedicated algorithms, but here we start with Groebner basis algorithms

Groebner Basis Experiments

- Studied comparable size systems in binary and prime cases
- Measured average values of degree of regularity
- Compared with semi-generic systems



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- Open problem !

Conclusion

- Suggested an approach to generalize previous ECDLP algorithms to elliptic curves over prime fields
- Like previous ones, algorithm only practical for very small parameters (Pollard's rho definitely better for crypto sizes)
- Open problems : asymptotic complexity, dedicated polynomial system solving methods