SQIsignHD

Sqiing in higher dimensions

Benjamin Wesolowski, CNRS and ENS de Lyon

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Based on joint works with **Andrea Basso**, **Pierrick Dartois**, **Luca De Feo**, **Antonin Leroux**, **Luciano Maino**, **Giacomo Pope**, and **Damien Robert**

SQIsign & friends

Isogeny-based signature schemes

Picture by Beppe Rijs

SQIsign

[De Feo, Kohel, Leroux, Petit, W. — Asiacrypt 2020] SQISign: compact post-quantum signatures from quaternions and isogenies

- **Isogeny-based** post-quantum signature scheme
- **Very compact**: PK + Signature combined **5× smaller** than Falcon

Secret key (bytes)	Public key (bytes)	Signature (bytes)	Security
16	64	204	NIST-I

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Original SQIsign	2800	4600	93
Optimized SQIsign	400	1880	29

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Drawbacks of SQIsign

- Signing in 600ms is **too slow**
- Security proof: the ZK property is based on an ***ad hoc* assumption**
- **Bad scaling** to higher security levels (signing at NIST-V takes 40s)

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Verification gets slower… Problem solved with

SQIsign2D

SQIsign2D-West (this talk) [Basso, Dartois, De Feo, Leroux, Maino, Pope, Robert, W. — Asiacrypt 2024]
SQIsign2D-East [Nakagawa, Onuki, Castryck, Chen, Invernizzi, Lorenzon, Vercauteren — Asiacrypt 2024]

The Isogeny problem

and how to represent an isogeny

Picture by Beppe Rijs

Elliptic curves

- In crypto, we use elliptic curves over a **finite field**
- Elliptic curves are **groups**: you can add points together!

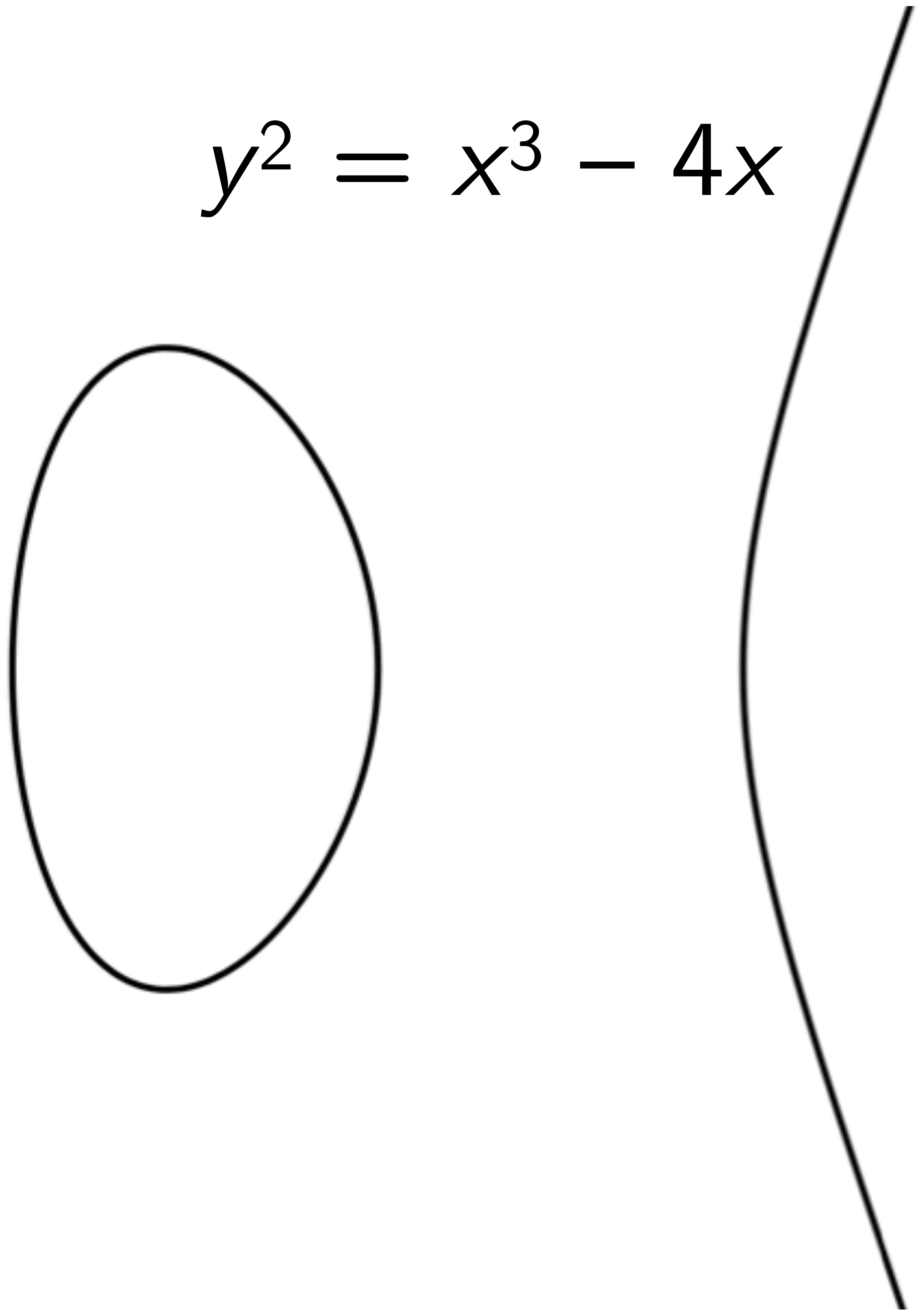
equations of the form

$$y^2 = x^3 + ax + b$$

$y^2 = x^3 + x$

E_1

$y^2 = x^3 - 4x$



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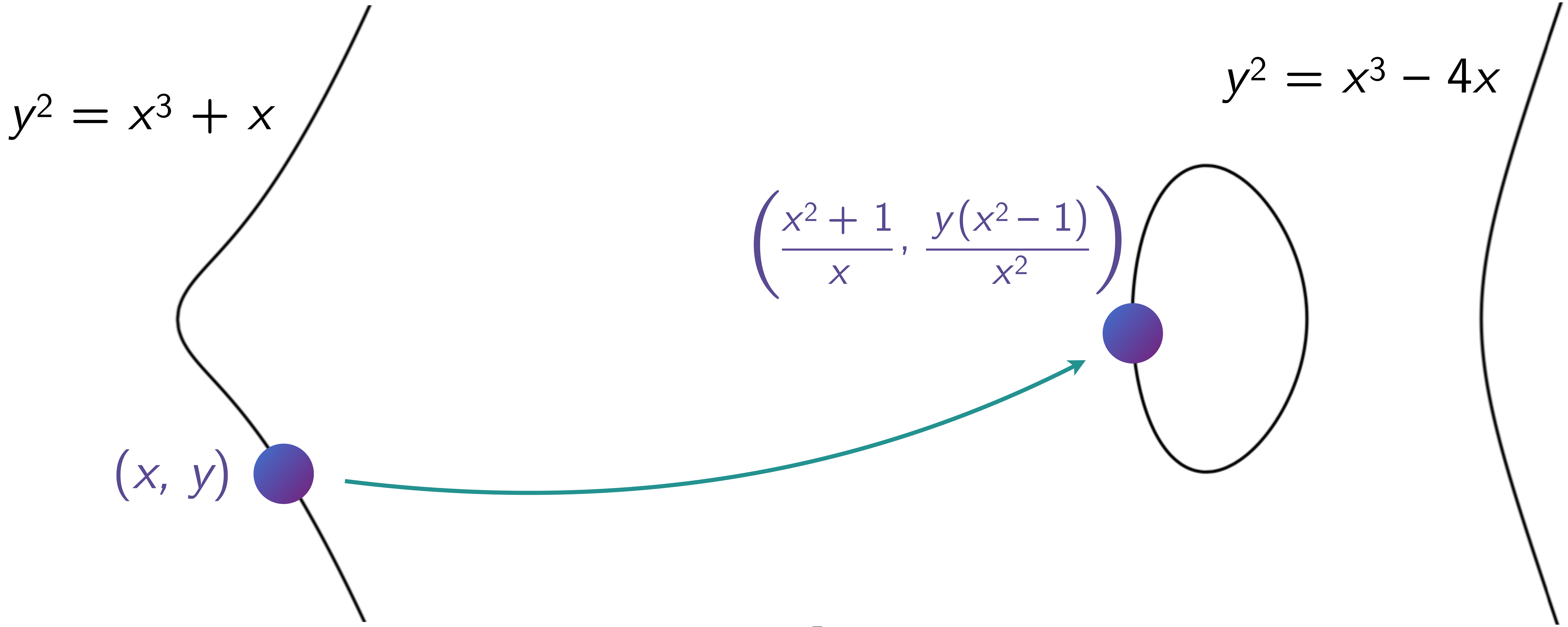
Elliptic curves

Sometimes, there is a formula to transform solutions from one equation to another



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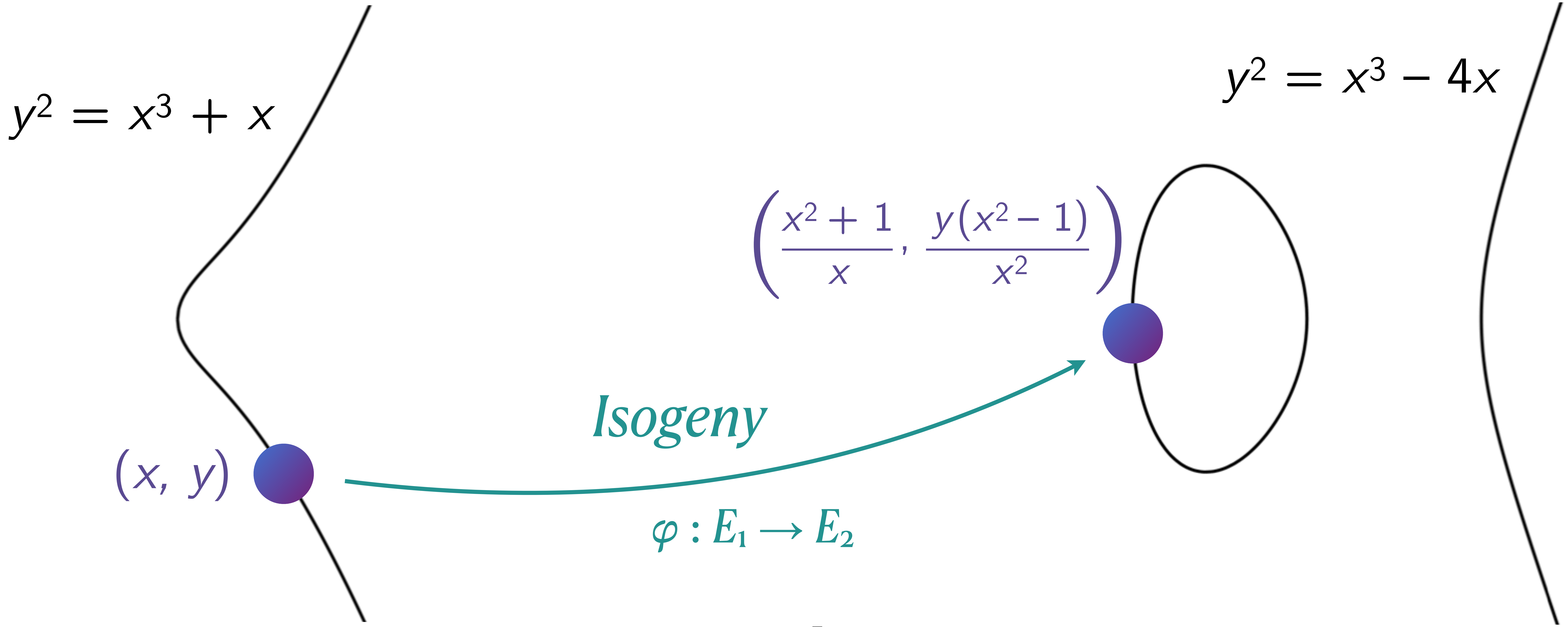

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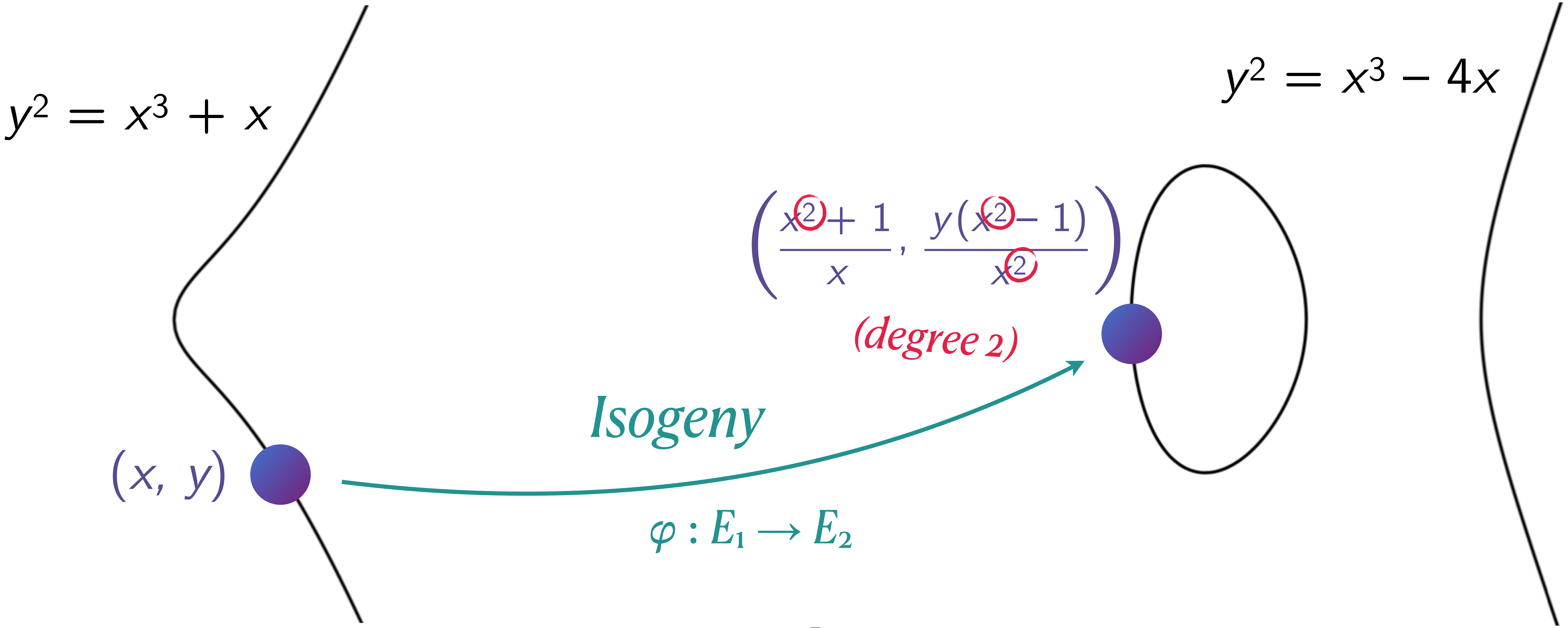
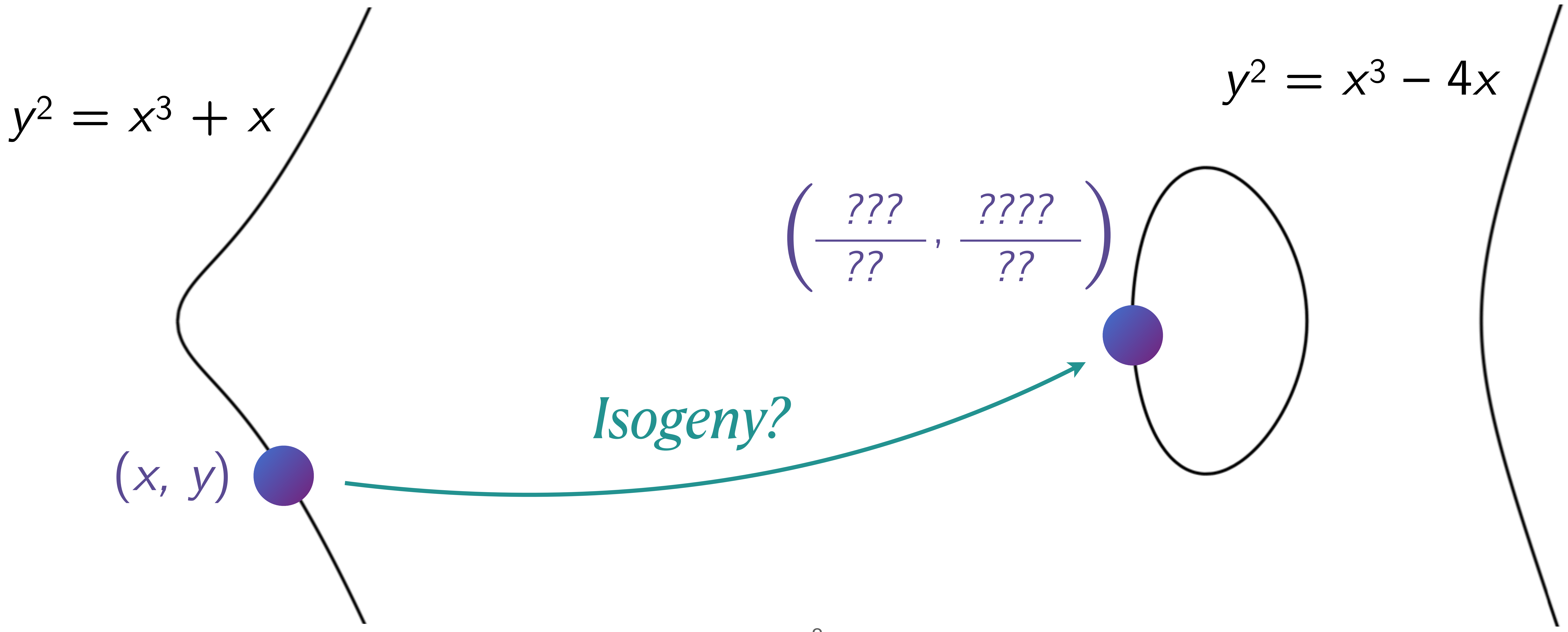


The *Isogeny* problem

Given E_1 and E_2 find an isogeny $\varphi : E_1 \to E_2$



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fine for small degree...

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$$\deg(\varphi \circ \psi) = \deg(\varphi) \cdot \deg(\psi)$$

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$$E_1 \xrightarrow{2} E_2 \xrightarrow{2} E_3 \xrightarrow{2} \dots \xrightarrow{2} E_{257}$$

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$$\begin{tikzcd} E_1 \arrow[r, "2"] \arrow[rrrr, bend right, "\text{degree } 2^{256}"'] & E_2 \arrow[r, "2"] & E_3 \arrow[r, "2"] & \ldots \arrow[r, "2"] & E_{257} \end{tikzcd}$$

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- Build "big" isogenies as *formal combinations* of "small" ones
  - $\varphi \circ \psi$ represented by ($'\mathtt{comp}'$, φ, ψ) where φ and ψ are in efficient repr.
  - $\varphi + \psi$ represented by ($'\mathtt{add}'$, φ, ψ) where φ and ψ are both $E_1 \to E_2$

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- How to represent an isogeny?
  - any **efficient representation**: an encoding which allows one to evaluate $\varphi(P)$ in polynomial time for any P

Isogeny graph

$$E_1 \longrightarrow E_2$$

an isogeny of degree 2 = an edge in a graph

Isogeny graph

- **The 2-isogeny graph** (supersingular…)

Isogeny graph

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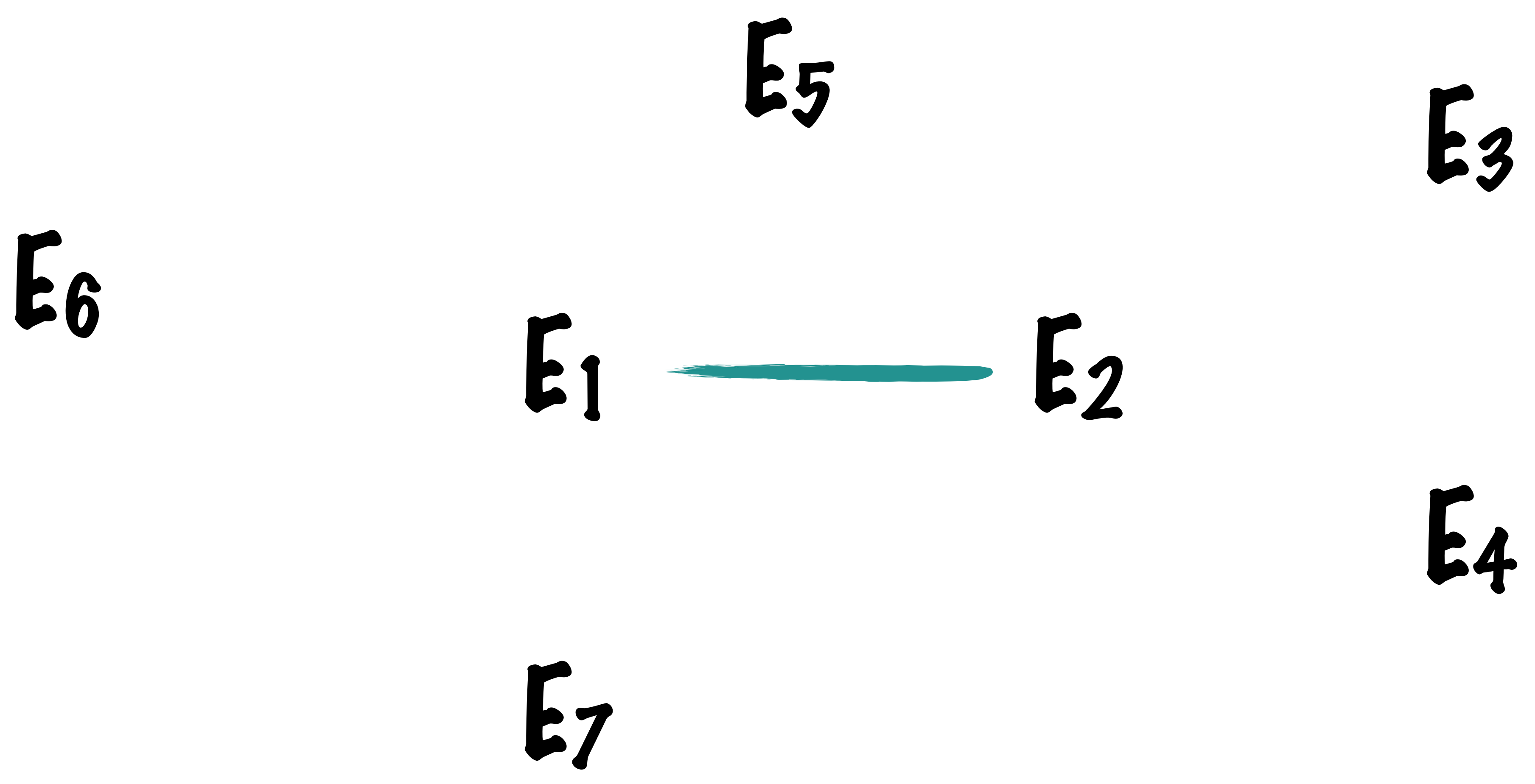

Isogeny graph

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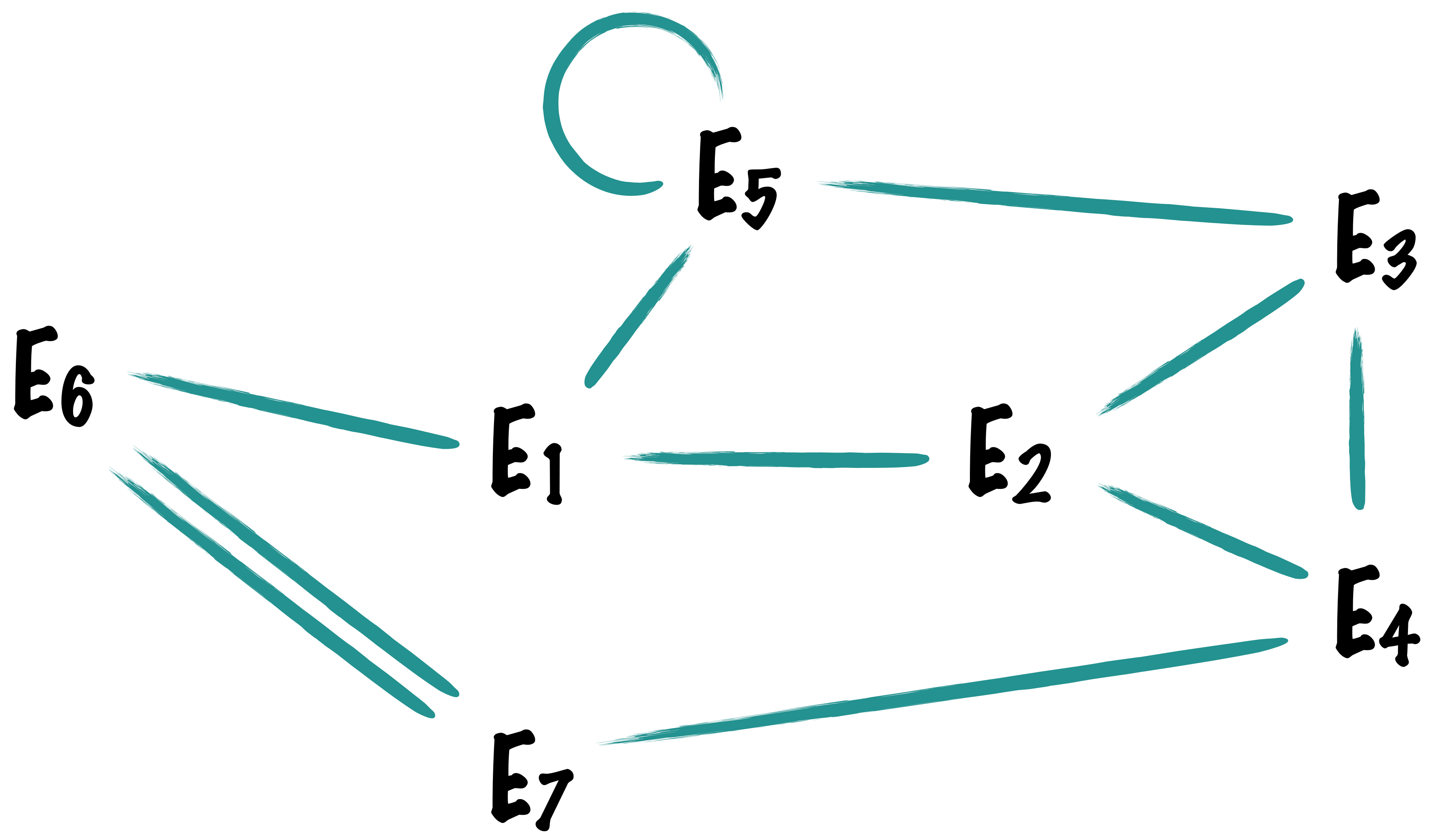

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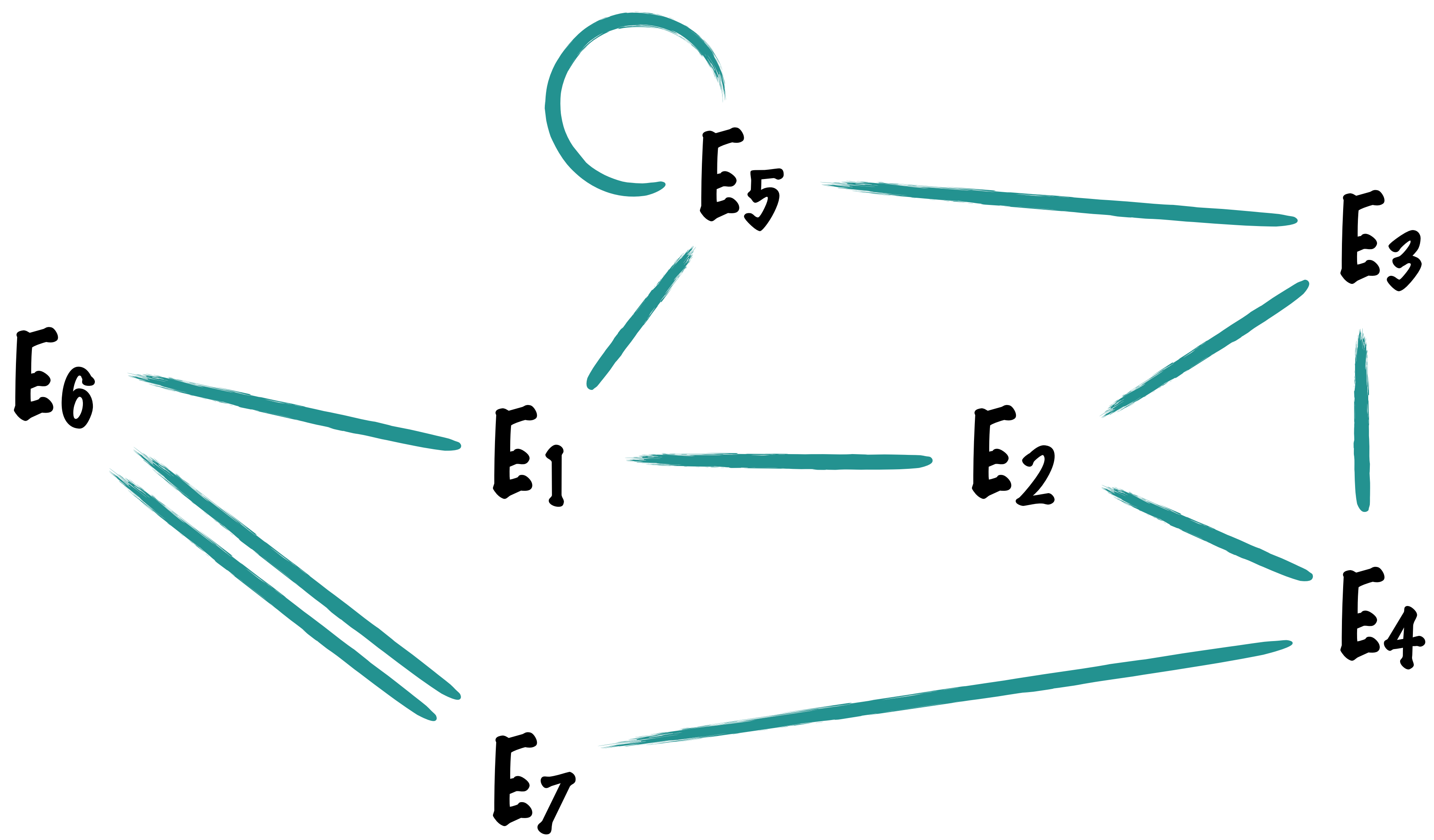

- 3-regular, **connected** (for supersingular curves)


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Uniformly distributed

Endomorphisms

and computational problems

Picture by Beppe Rijs

Endomorphism ring

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Multiplication by $m \in \mathbb{Z}$ is an endomorphism

$$[m] : E \to E : P \mapsto P + \ldots + P$$

It forms a subring $\mathbb{Z} \subset \mathrm{End}(E)$

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- It contains $\mathbb{Z} \subset \mathrm{End}(E)$...

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A curve E is **supersingular** if (End(E), +) is a lattice of dimension 4

Then, there is a $\mathbb{Z}$-basis 1, α_2, α_3, α_4: as a lattice,

$$\text{End}(E) = \mathbb{Z} \oplus \mathbb{Z}\alpha_2 \oplus \mathbb{Z}\alpha_3 \oplus \mathbb{Z}\alpha_4$$

The endomorphism ring problem

For E **supersingular** $\mathrm{End}(E) = \{\varphi : E \to E\}$ is a lattice of dimension 4

The *EndRing* problem

Given E (supersingular) find 4 generators of the endomorphism ring $\mathrm{End}(E)$

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Solution of the form $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$...

- How are α_i represented? Any *efficient representation*

Example

Example: $p \equiv 3 \pmod 4$, so $\mathbb{F}_{p^2} = \mathbb{F}_p(\alpha)$ where $\alpha^2 = -1$, and

Consider $E_0 : y^2 = x^3 + x$

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EndRing

EndRing problem ⇔ **Isogeny** problem

Computing End(–) $\Leftrightarrow$ Finding isogenies

[Petit, Lauter – preprint 2017] *Hard and Easy Problems for Supersingular Isogeny Graphs*.

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Computing End(–) ⇔ Computing Hom(–, –)

Key generation

Generating a curve with its endomorphism ring

Picture by Beppe Rijs

Idea of SQIsign

Basic idea of SQIsign:

- **Public key**: a supersingular curve E_{pk}
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Key recovery = EndRing

How to generate a random E_{pk} together with $\text{End}(E_{\text{pk}})$?

How to generate even a single supersingular curve?

A special supersingular curve

Example: $p \equiv 3 \pmod 4$, so $\mathbb{F}_{p^2} = \mathbb{F}_p(\alpha)$ where $\alpha^2 = -1$, and

Consider $\boldsymbol{E_0 : y^2 = x^3 + x}$

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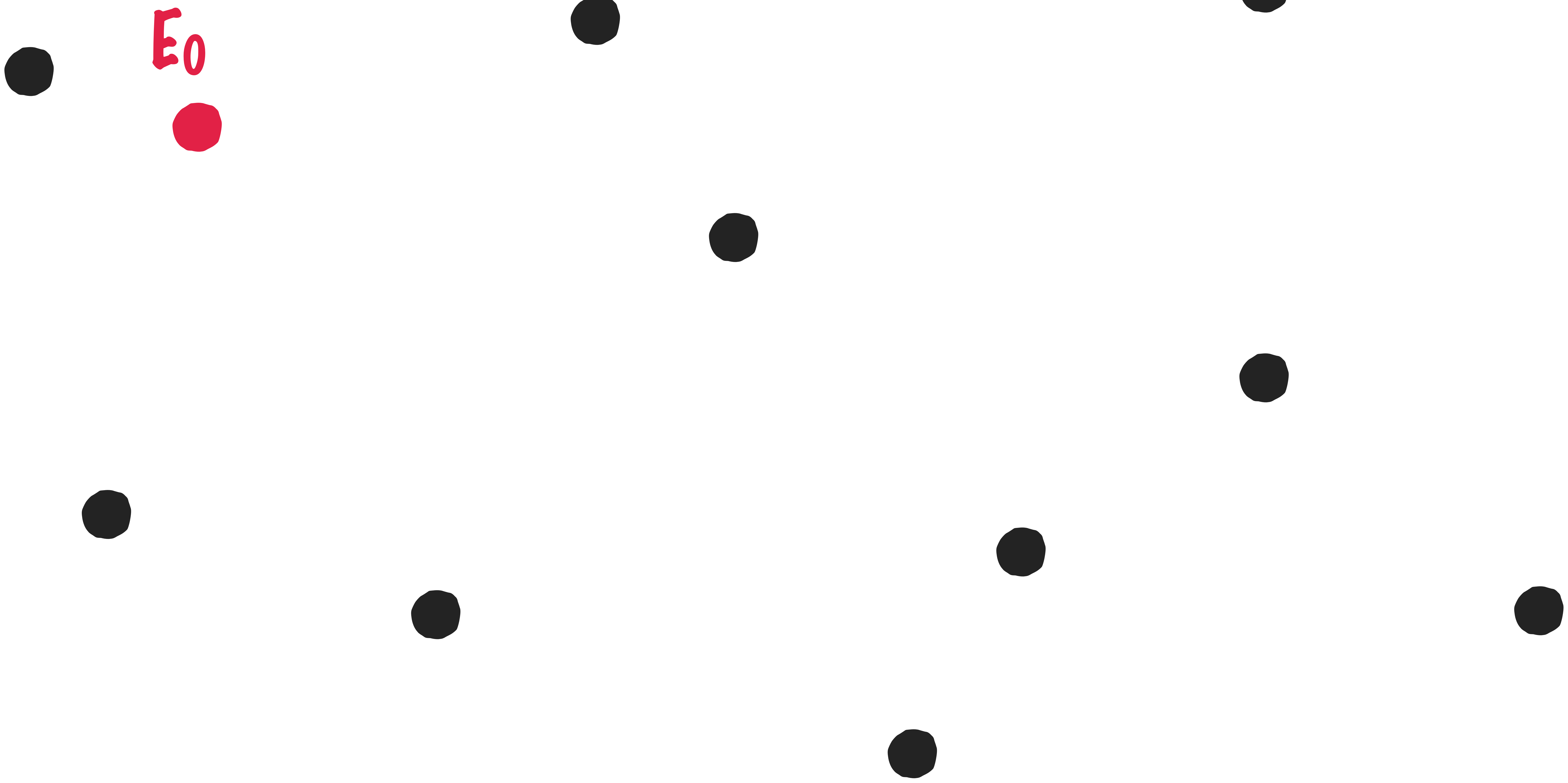
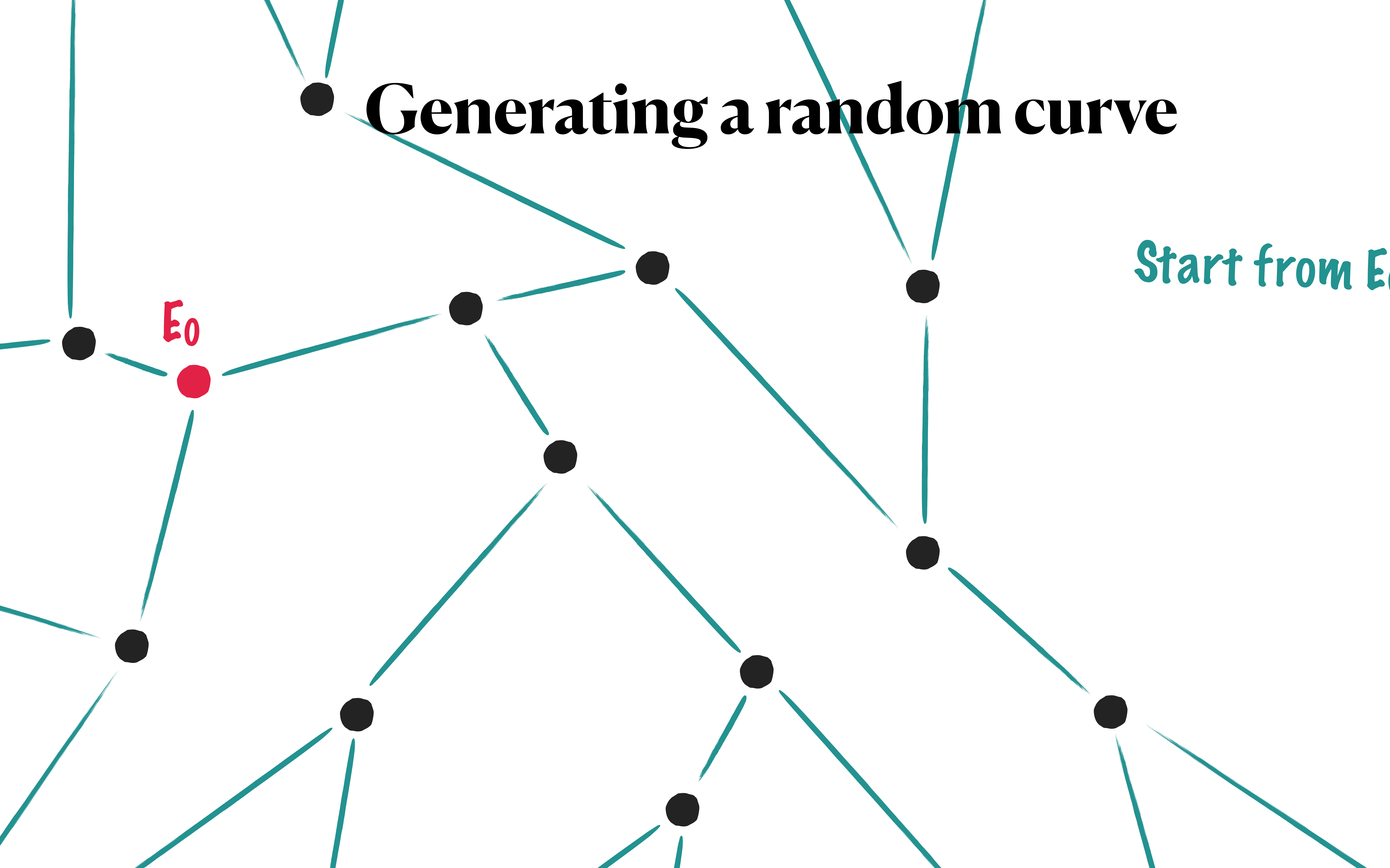
E_0 and $\mathrm{End}(E_0)$ is our reference

- # Generating a random curve



Start from E_0

Generating a random curve



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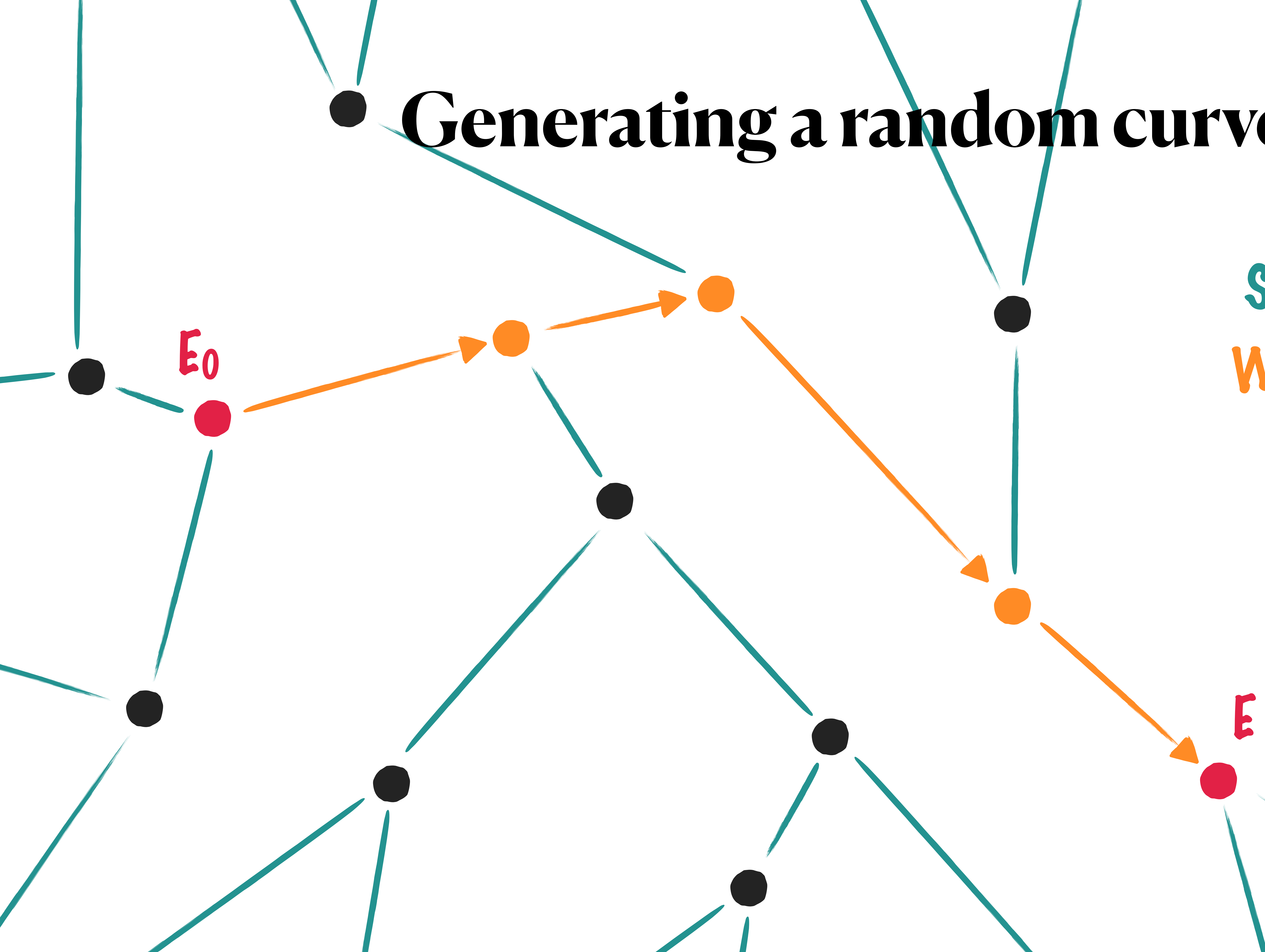
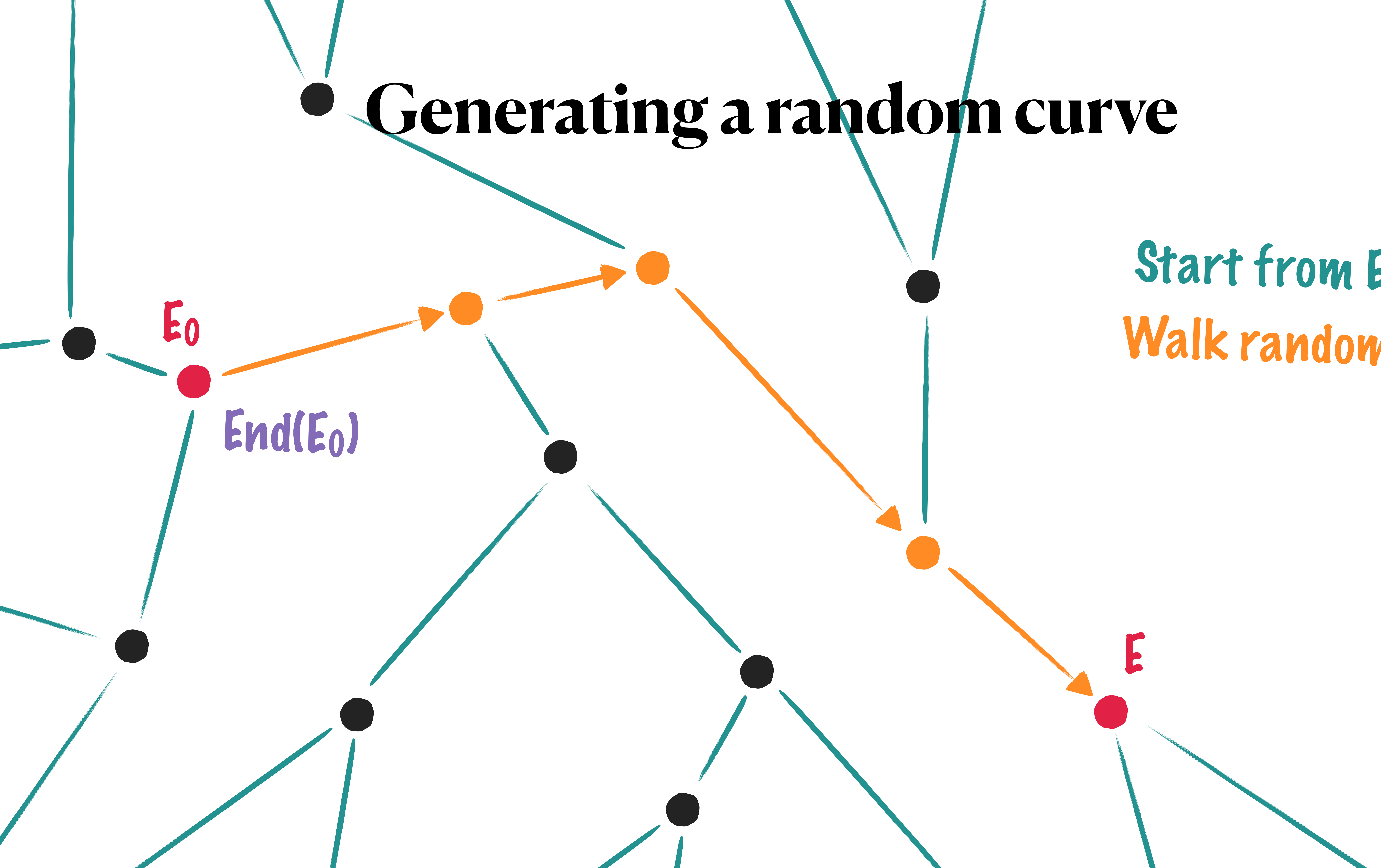


Start from E0

Walk randomly

Generating a random curve



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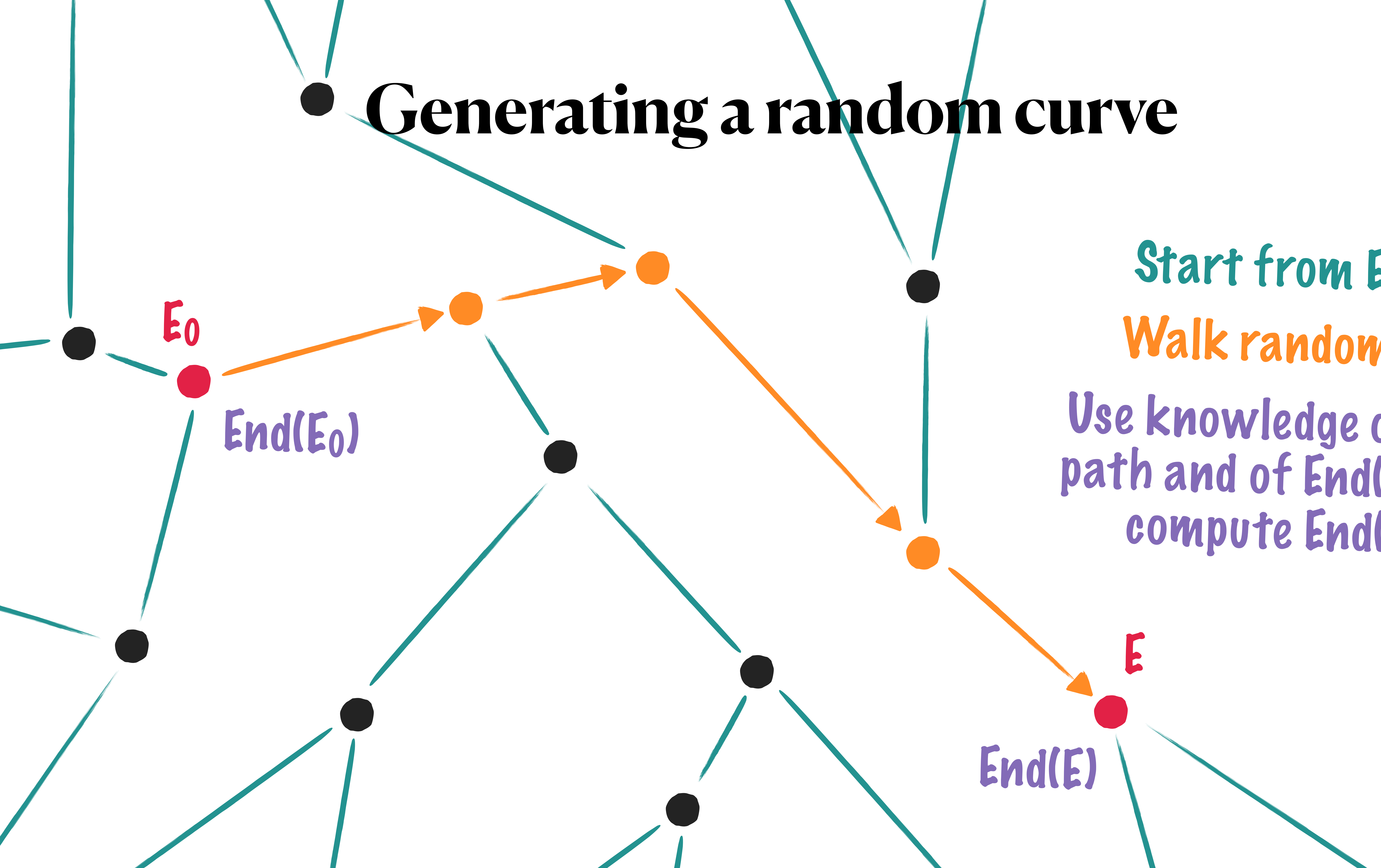

Start from E_0

Walk randomly

Use knowledge of the path and of $\mathrm{End}(E_0)$ to compute $\mathrm{End}(E)$

One can generate $(E, \mathrm{End}(E))$ with E uniform

One can generate $(E, \mathrm{End}(E))$ with E uniform

(Trapdoor generation of uniform **EndRing** instances)

SQIsign

Proving knowledge of an endomorphism

Picture by Beppe Rijs

A sigma protocol following [DKLPW20]

- Generate a random pair (E_{pk}, $\mathrm{End}(E_{\mathrm{pk}})$)
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Bob (verifier)

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End(E_{pk})

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Generate random (E_{com}, $\text{End}(E_{\text{com}})$) $\xrightarrow{E_{\text{com}}}$

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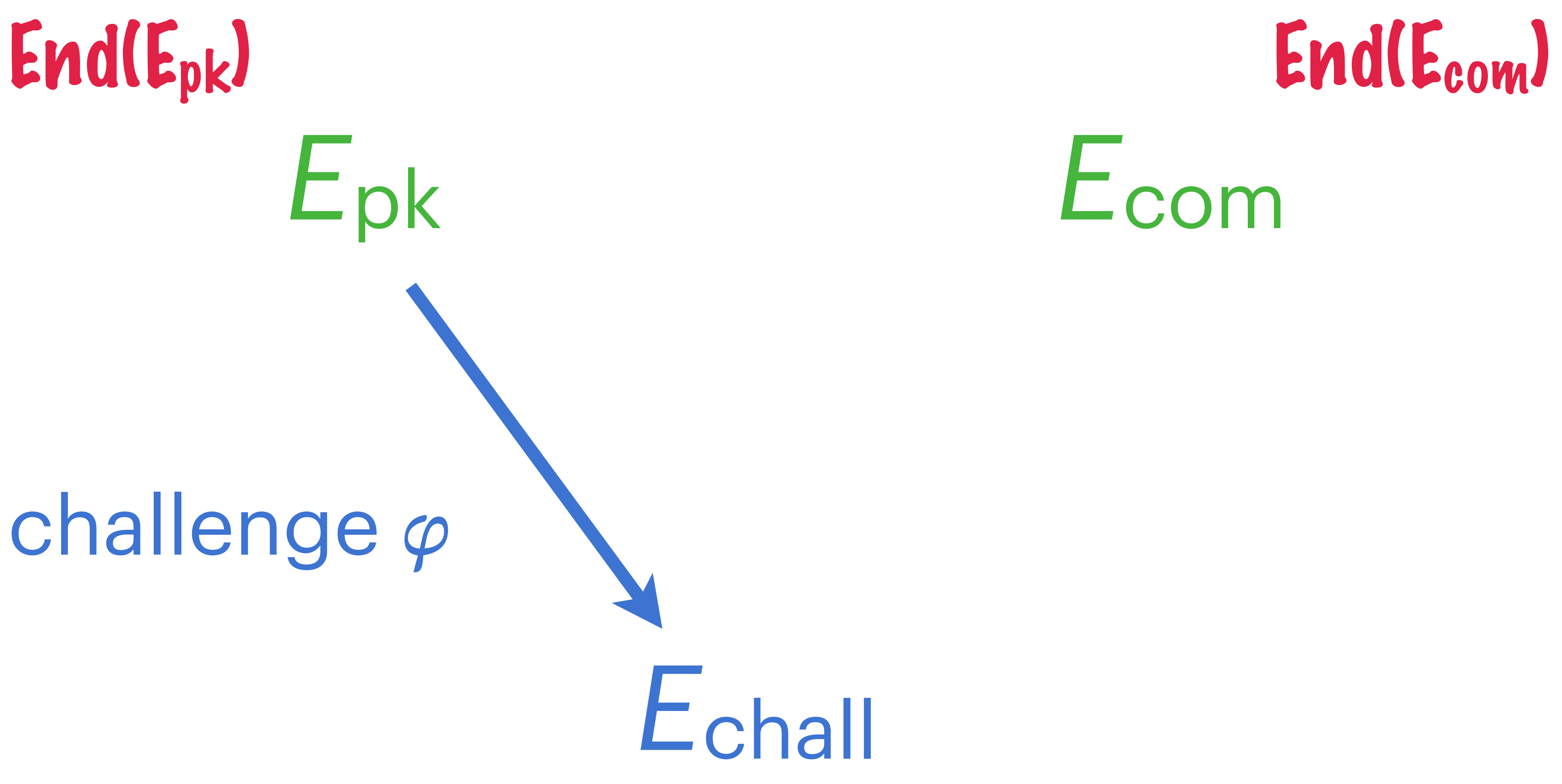

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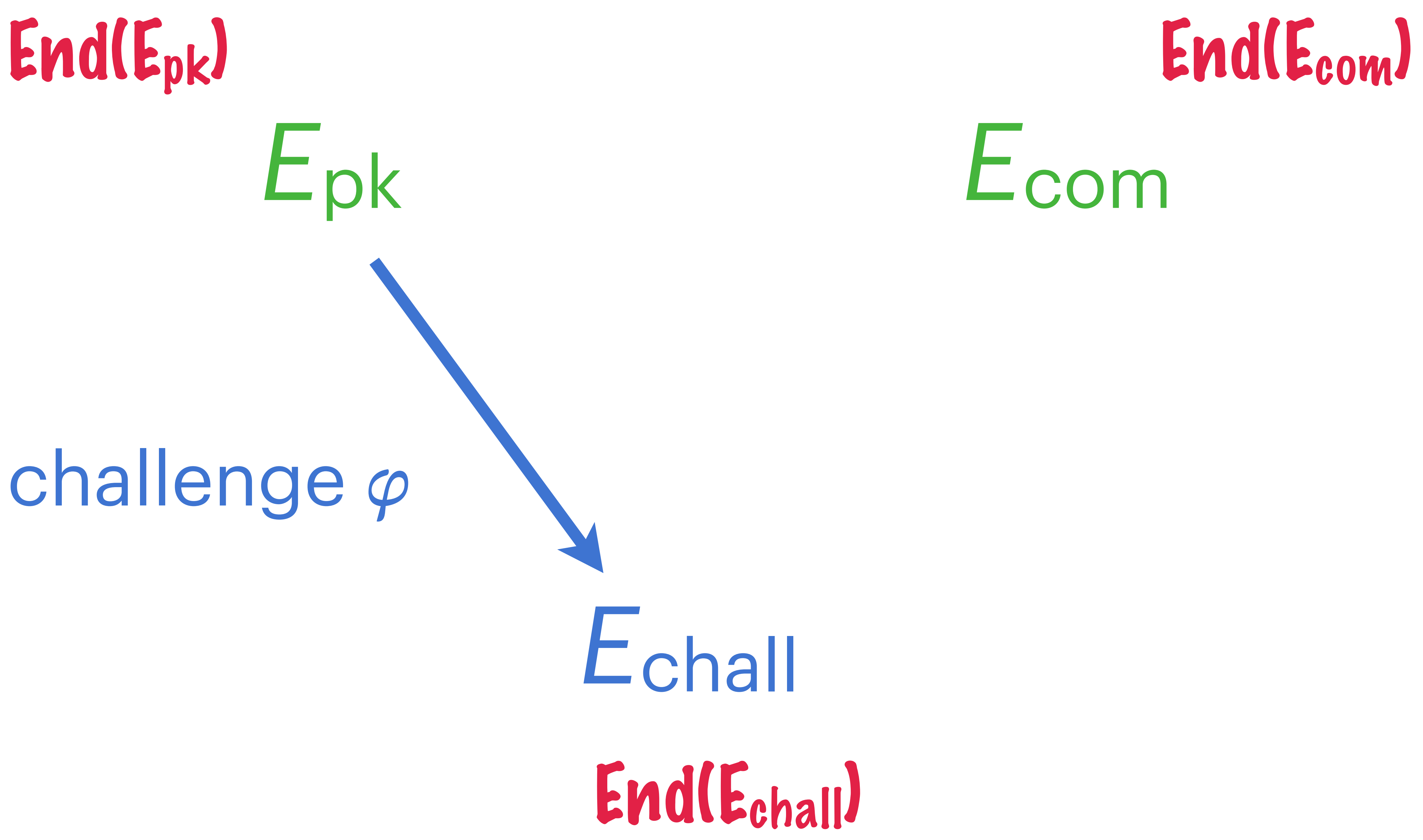

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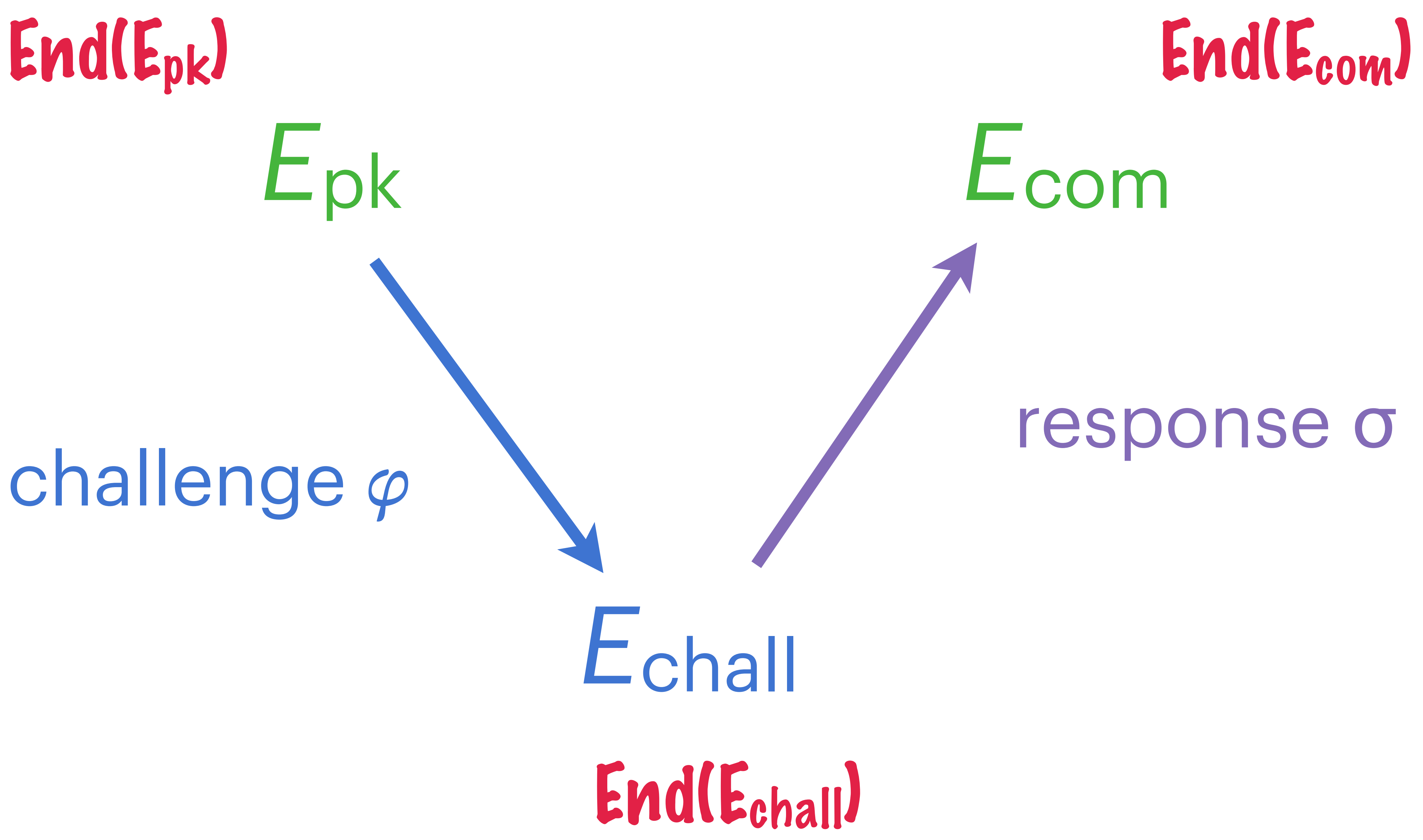

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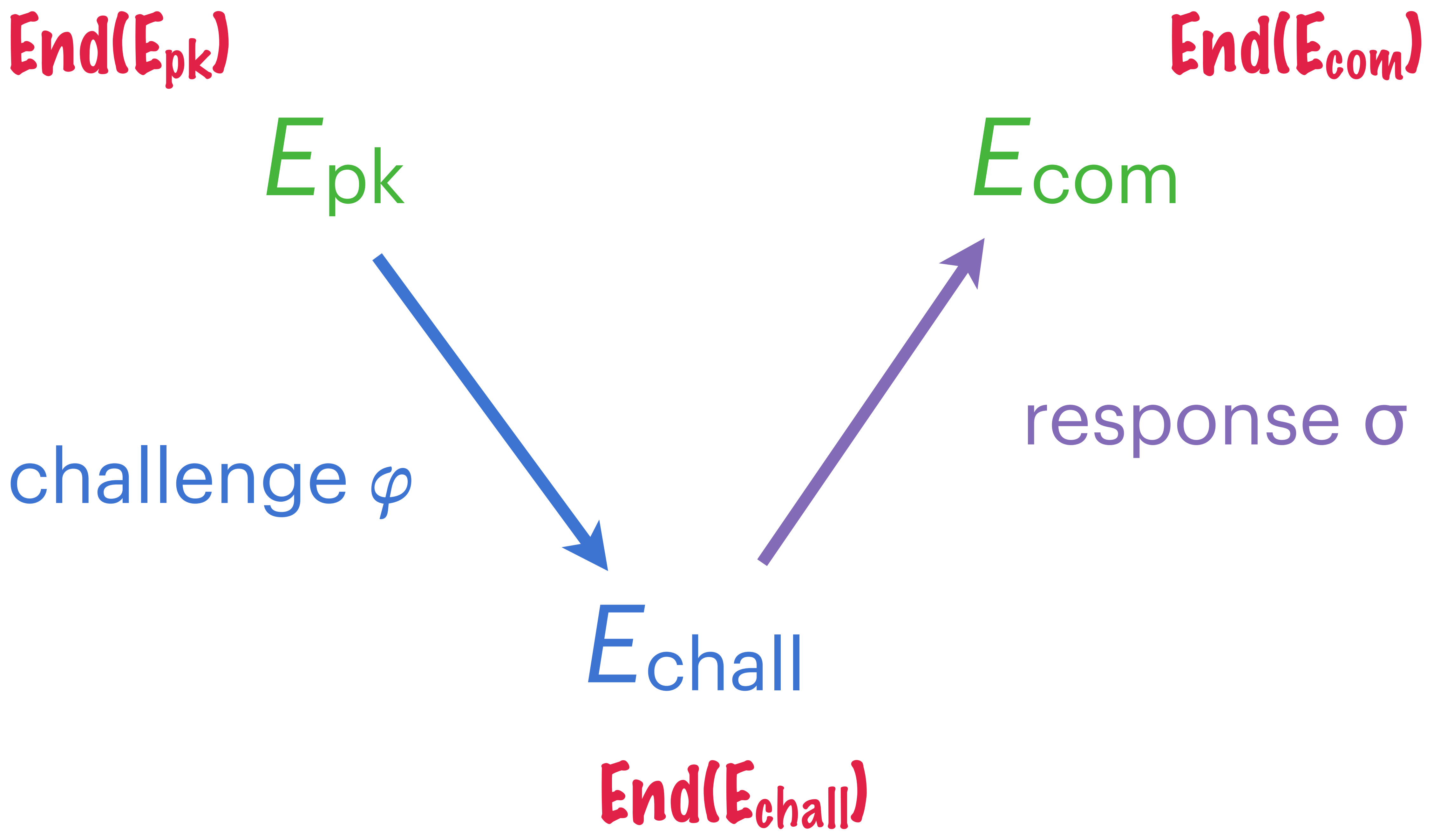

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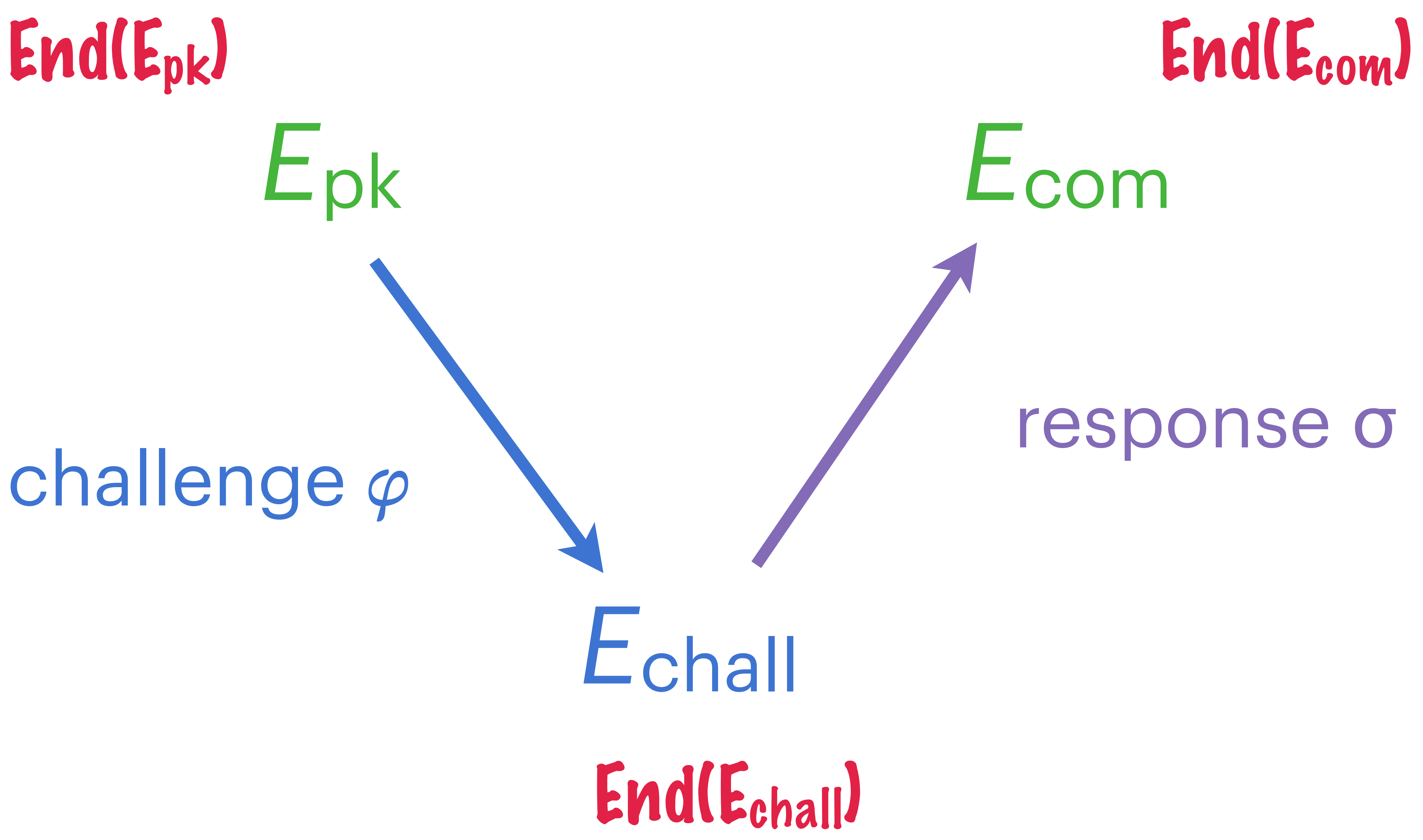

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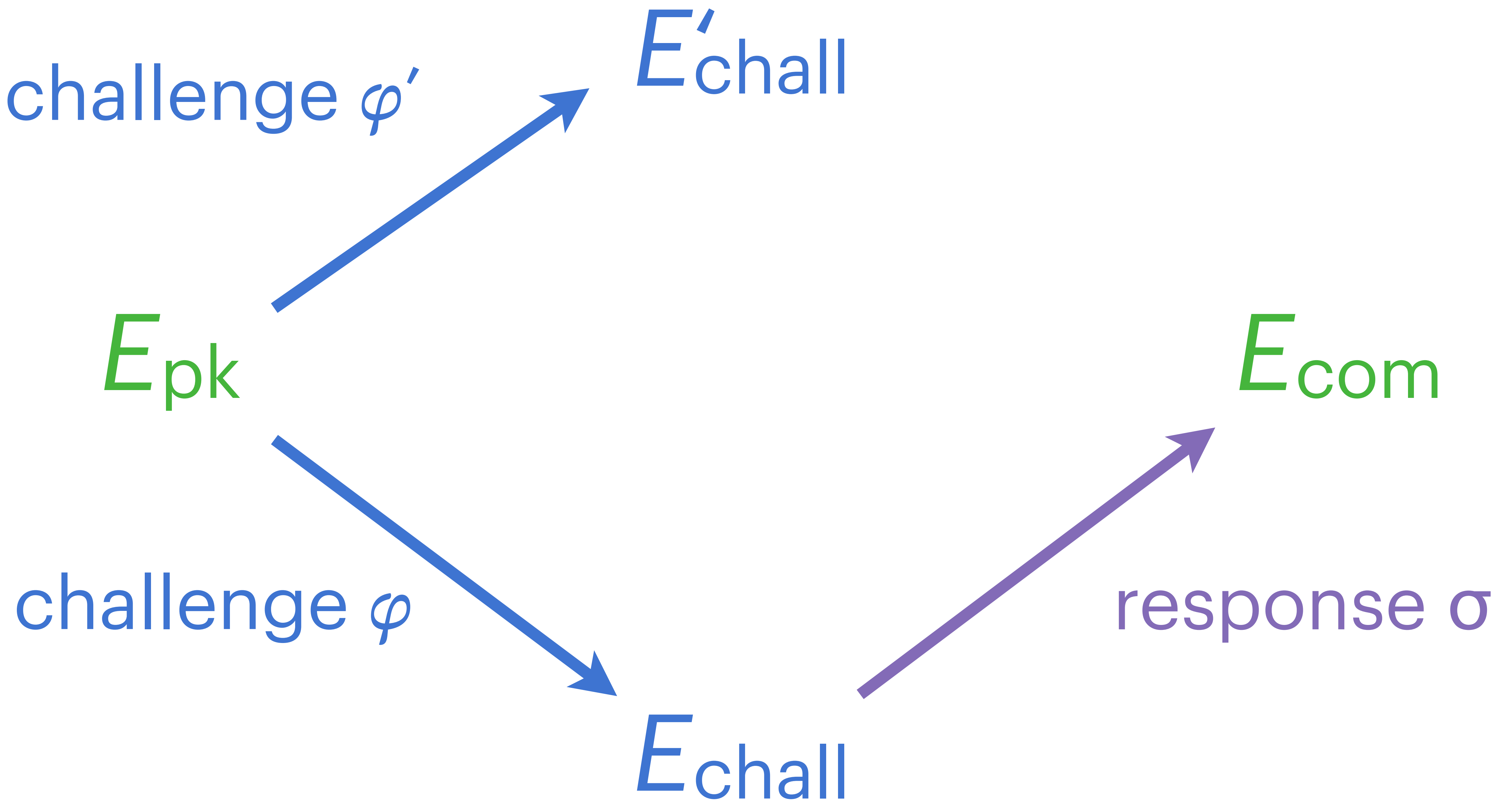
Compute $\sigma : E_{\text{chall}} \to E_{\text{com}}$	$\xrightarrow{\sigma}$	Check that σ is an isogeny $E_{\text{chall}} \to E_{\text{com}}$

Special soundness

$$E_{\mathrm{pk}} \xrightarrow{\text{challenge } \varphi} E_{\mathrm{chall}} \xrightarrow{\text{response } \sigma} E_{\mathrm{com}}$$

Special soundness



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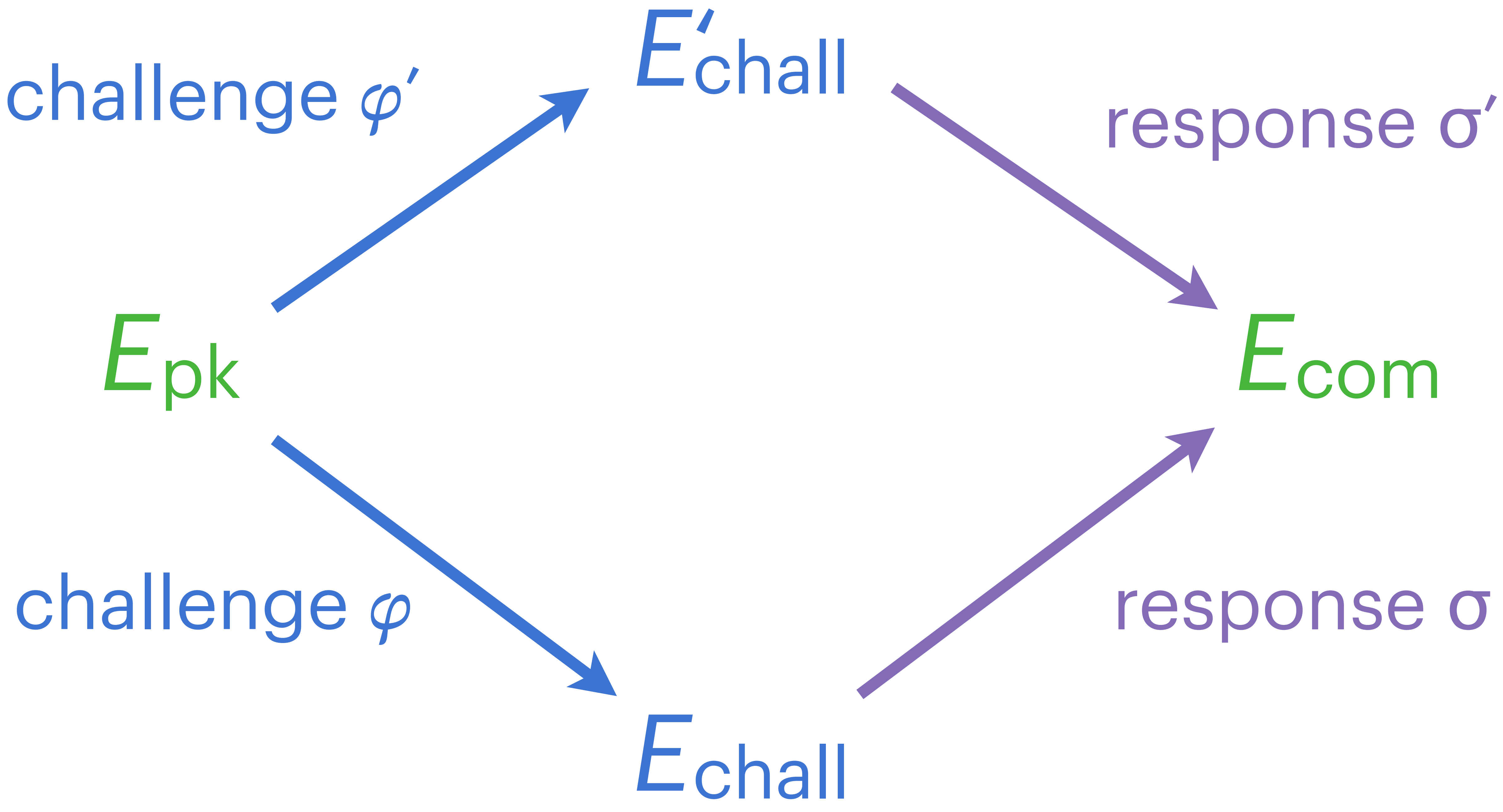

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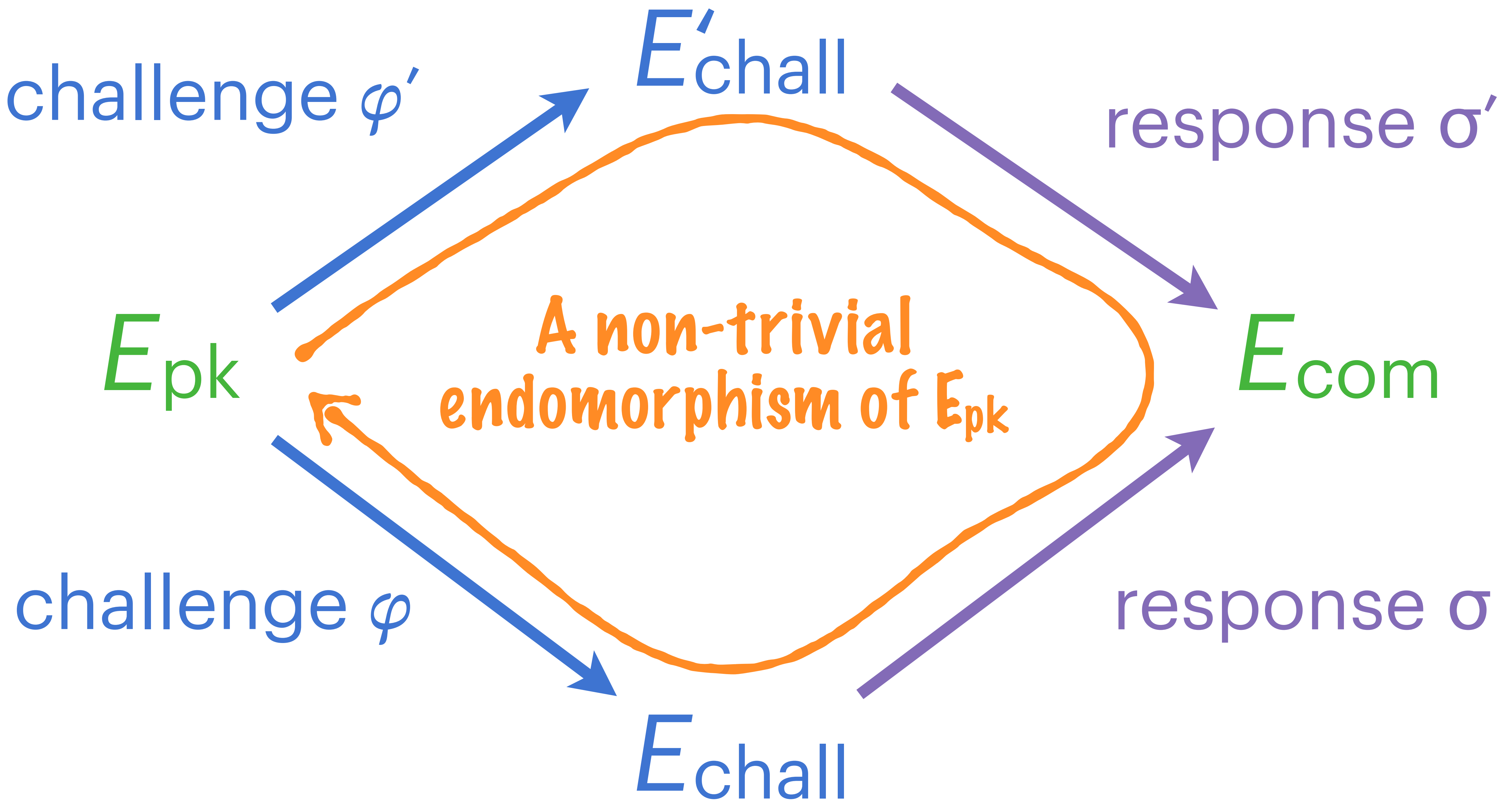

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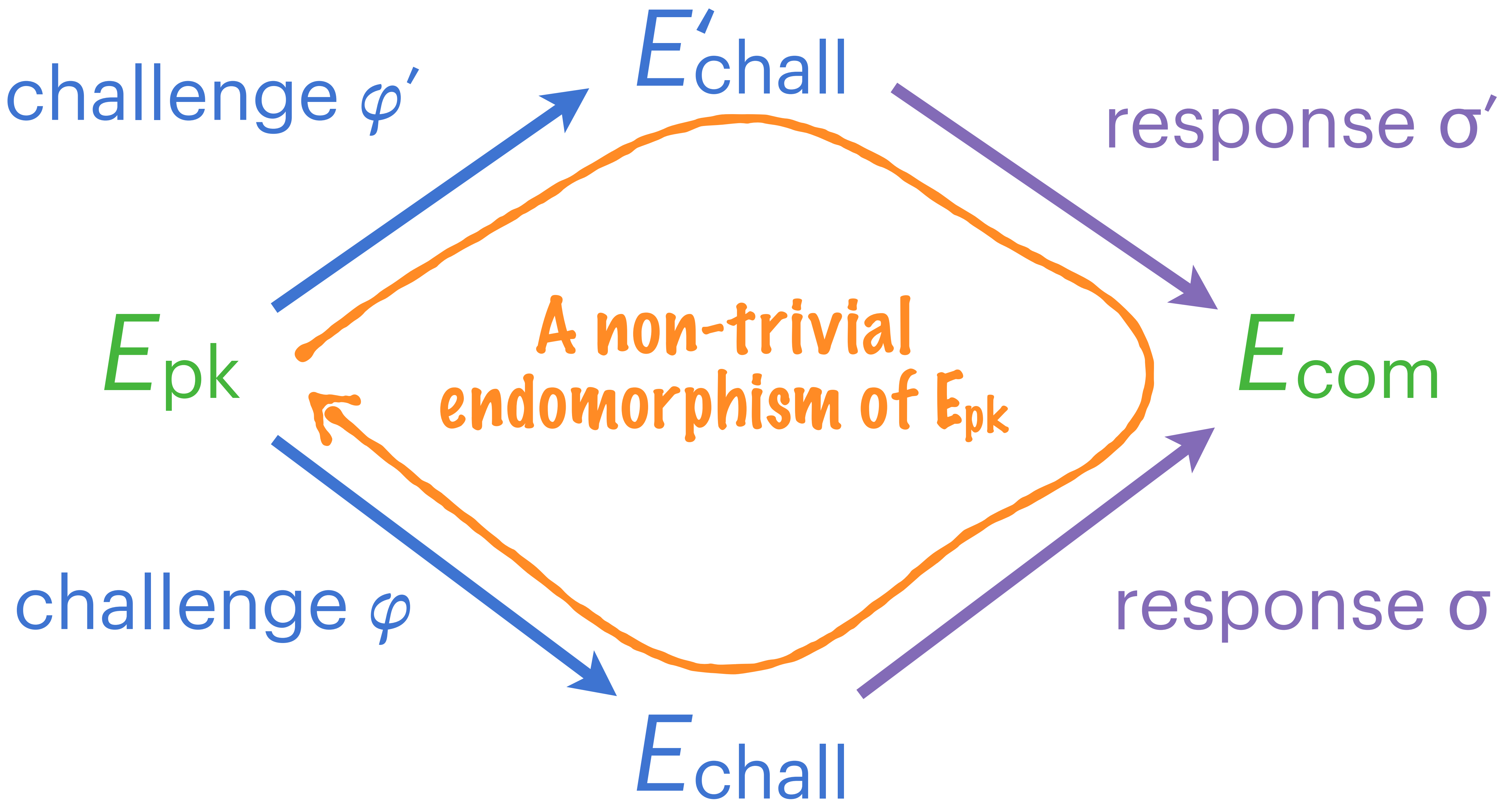
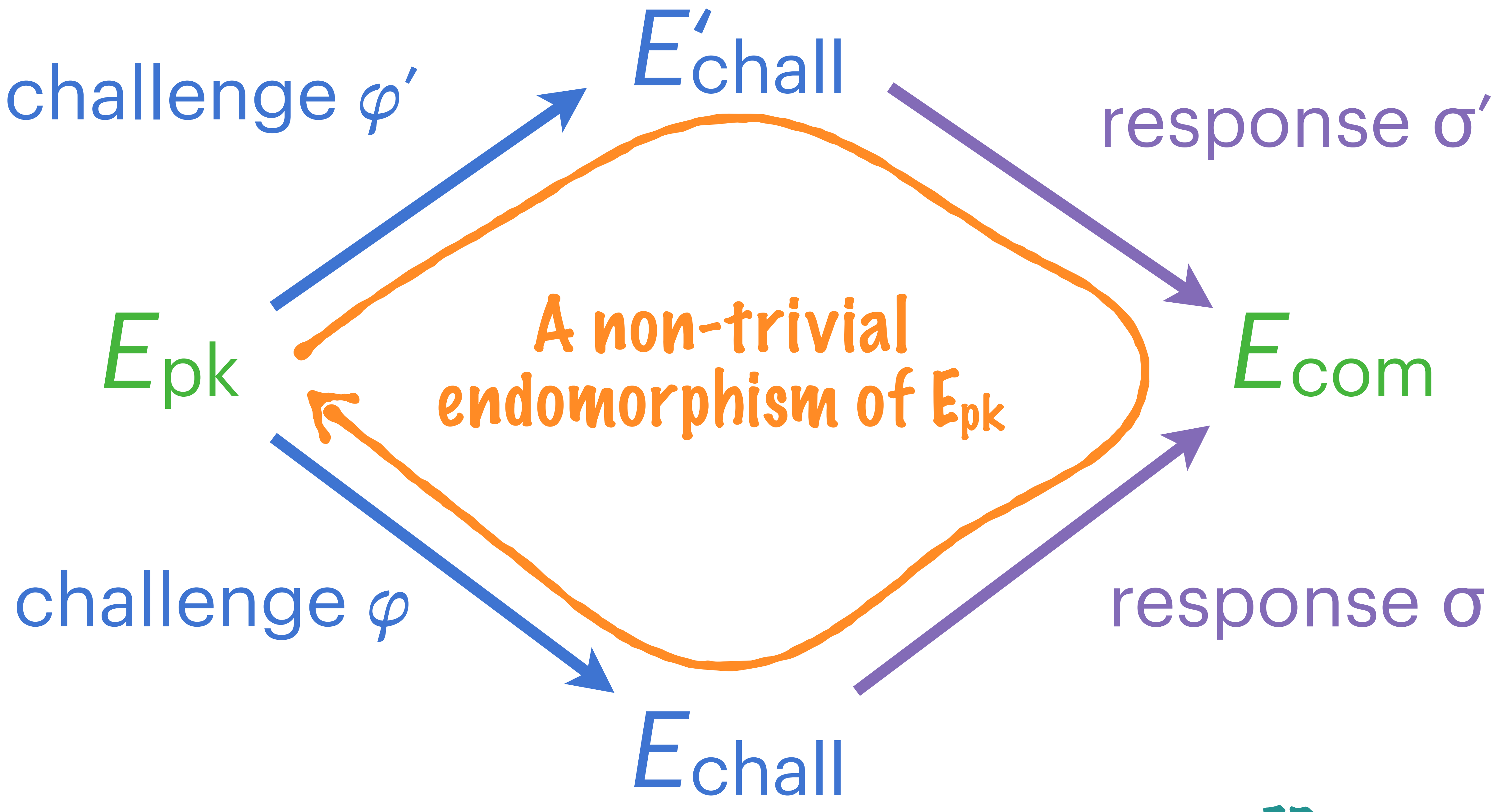


Can respond to 2 challenges ⇒ **Can find an endomorphism**

Special soundness



[Page, W. - Eurocrypt 2024]
Finding one endomorphism ⇔ EndRing

Can respond to 2 challenges ⇒ **Can find an endomorphism**

Computing the response isogeny

$\mathrm{End}(E_{\mathrm{chall}})$ $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $\mathrm{End}(E_{\mathrm{com}})$

$$E_{\mathrm{chall}} \xrightarrow{\text{response } \sigma} E_{\mathrm{com}}$$

Computing the response isogeny

$\mathrm{End}(E_{\mathrm{chall}})$ $\qquad\qquad$ $\mathrm{End}(E_{\mathrm{com}})$

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Computing the response isogeny

End(E_{chall}) End(E_{com})

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2. Return some $\sigma \in$ Hom(E_{chall}, E_{com})

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Question: how to choose σ? How to represent it?

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- Need σ fast to evaluate, for efficient verification

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2. Return some $\sigma \in$ Hom(E_{chall}, E_{com})

Question: how to choose σ? How to represent it?

- Need σ fast to evaluate, for efficient verification
- Need σ (and its representation) not to leak the secret
  - **Warning:** typical representation of φ_i leaks End(E_{chall})

Computing the response isogeny

$$\underset{\mathrm{End}(E_{\mathrm{chall}})}{E_{\mathrm{chall}}} \xrightarrow{\text{response } \sigma} \underset{\mathrm{End}(E_{\mathrm{com}})}{E_{\mathrm{com}}}$$

Original SQIsign [DKLPW20]**:**

- Solve a norm equation [KLPT14] to find $\sigma \in \mathrm{Hom}(E_{\mathrm{chall}}, E_{\mathrm{com}})$ of degree 2^n,
- Convert $\sigma = a_1\varphi_1 + a_2\varphi_2 + a_3\varphi_3 + a_4\varphi_4$ to path of 2-isogenies

Computing the response isogeny

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Problems:

- [KLPT14] finds **big** solution: $\deg(\sigma) = 2^n \approx p^{3.75}$
- Conversion to chain of 2-isogenies is very **costly**: signing takes billions of cycles
- Distribution of [KLPT14] output is mysterious: **not simulatable**? not zero-knowledge?

SQIsignHD

Sqiing in higher dimensions

Picture by Beppe Rijs

Computing the response isogeny

$\mathrm{End}(E_{chall})$ $\qquad\qquad\qquad\qquad$ $\mathrm{End}(E_{com})$

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Computing the response isogeny

$$\overset{\mathrm{End}(E_{chall})}{E_{chall}} \xrightarrow{\text{response } \sigma} \overset{\mathrm{End}(E_{com})}{E_{com}}$$

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- Smallest isogeny in $\mathrm{Hom}(E_{chall}, E_{com})$ has degree $\approx p^{0.5}$

Computing the response isogeny

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$$E_{\text{chall}} \xrightarrow{\text{response } \sigma} E_{\text{com}}$$

Original SQIsign [DKLPW20]**:** [KLPT14] finds big solution $\deg(\sigma) = 2^n \approx p^{3.75}$

- Smallest isogeny in $\mathrm{Hom}(E_{\text{chall}}, E_{\text{com}})$ has degree $\approx p^{0.5}$

Question: Why not output some small $\sigma \in \mathrm{Hom}(E_{\text{chall}}, E_{\text{com}})$, $\deg(\sigma) \approx p^{0.5}$?

Computing the response isogeny

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Original SQIsign [DKLPW20]**:** [KLPT14] finds big solution $\deg(\sigma) = 2^n \approx p^{3.75}$

- Smallest isogeny in $\mathrm{Hom}(E_{\mathrm{chall}}, E_{\mathrm{com}})$ has degree $\approx p^{0.5}$

Question: Why not output some small $\sigma \in \mathrm{Hom}(E_{\mathrm{chall}}, E_{\mathrm{com}})$, $\deg(\sigma) \approx p^{0.5}$?

- Representation as linear combination $a_1\varphi_1 + a_2\varphi_2 + a_3\varphi_3 + a_4\varphi_4$ is dangerous:
  - $(\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ leaks $\mathrm{End}(E_{\mathrm{chall}})$, so **leaks secret key** $\mathrm{End}(E_{\mathrm{pk}})$

HD representation of isogenies

Attacks against SIDH [CD23, MMPPW23, Rob23]**:**

HD representation of isogenies

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Very costly, but always works

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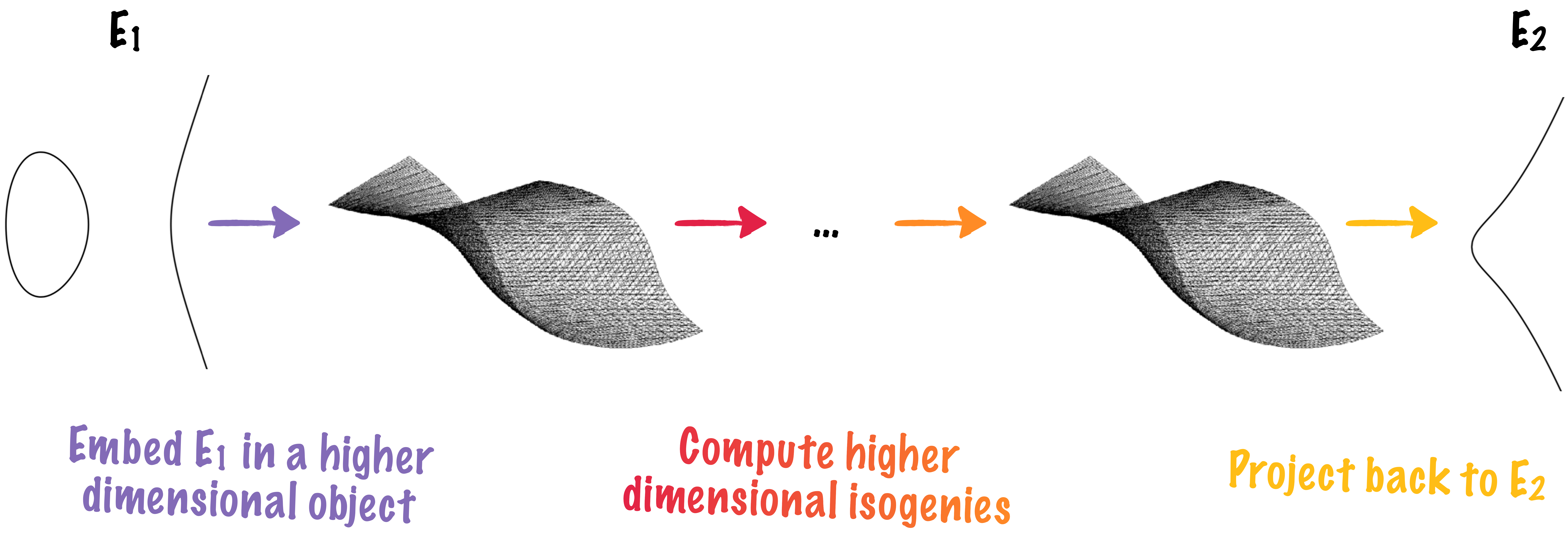
HD representation of isogenies



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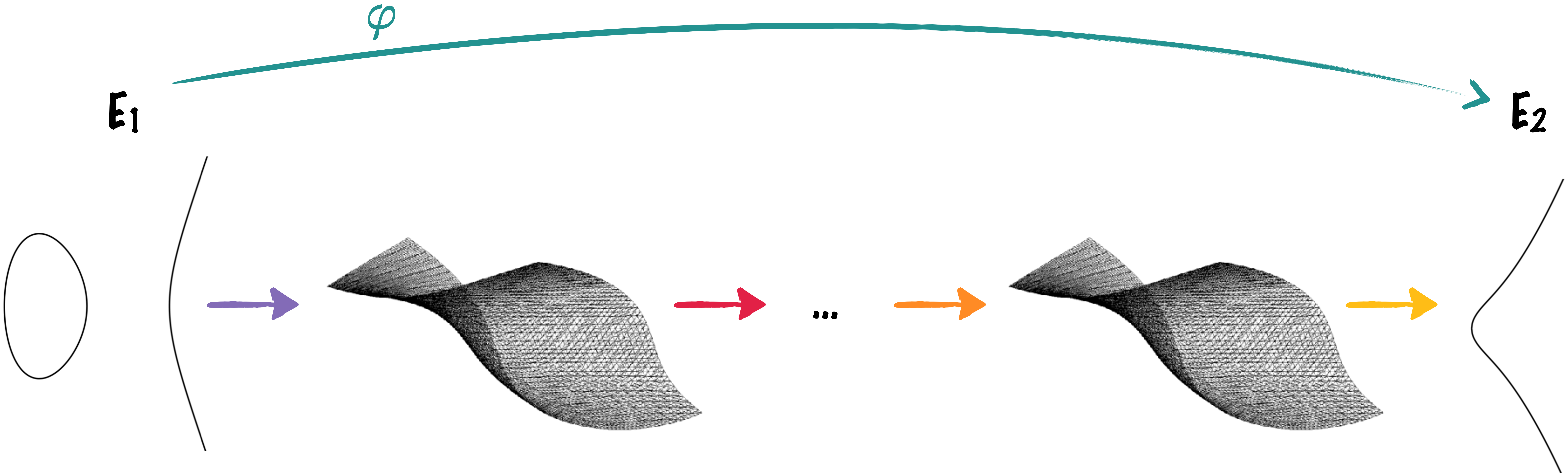
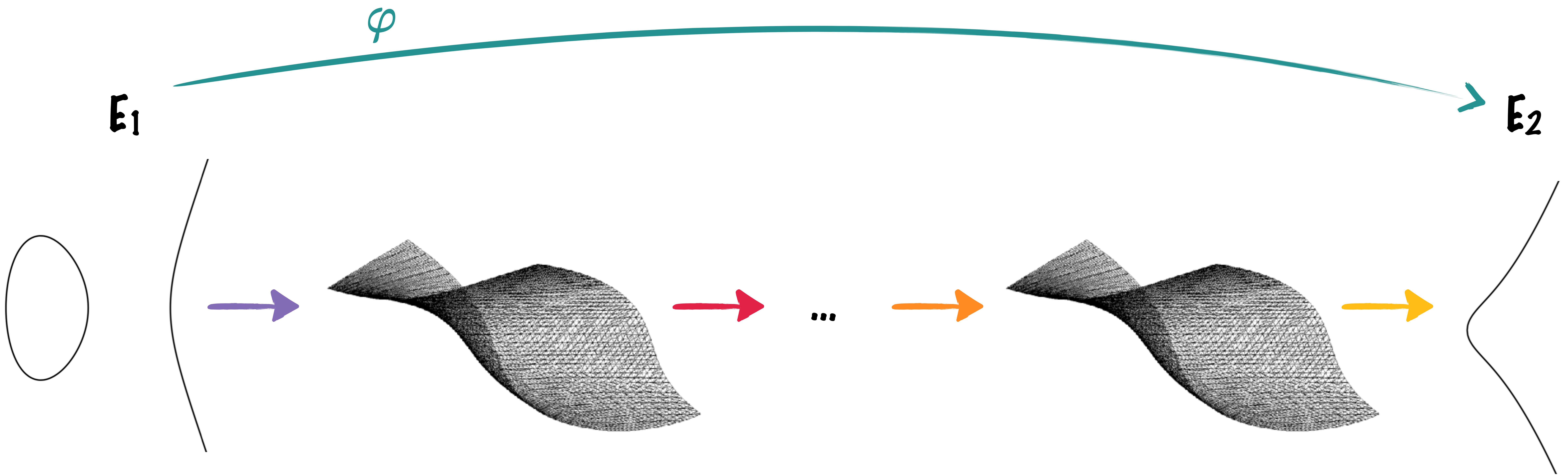


Embed E_1 in a higher dimensional object

Compute higher dimensional isogenies

Project back to E_2

HD representation of isogenies



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Determined by what φ does on $E_1[2^n]$

HD response isogeny

$\mathrm{End}(E_{chall})$ $\qquad$ $\mathrm{End}(E_{com})$

$$E_{\mathrm{chall}} \xrightarrow{\text{response } \sigma} E_{\mathrm{com}}$$

SQIsignHD [DLRW24]**:** constructive use of SIDH attacks

HD response isogeny

$$\begin{array}{ccc} \mathrm{End}(E_{chall}) & & \mathrm{End}(E_{com}) \\ E_{chall} & \xrightarrow{\text{response } \sigma} & E_{com} \end{array}$$

SQIsignHD [DLRW24]: constructive use of SIDH attacks

1. Pick random, **small** $\sigma \in \mathrm{Hom}(E_{chall}, E_{com})$ (say, $\deg(\sigma) \approx p^{0.5}$)

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3. Evaluate $P' = \sigma(P)$ and $Q' = \sigma(Q)$
4. $(\deg(\sigma), P, Q, P', Q')$ is an **HD representation** of σ

Zero-knowledge

From *ad hoc* to rigorous security proof

Picture by Beppe Rijs

Honest verifier zero-knowledge

Alice (prover)		**Bob** (verifier)
Generate random $(E_{\text{com}}, \text{End}(E_{\text{com}}))$	$\xrightarrow{E_{\text{com}}}$	
	$\xleftarrow{\varphi}$	Generate random $\varphi : E_{\text{pk}} \to E_{\text{chall}}$
Compute $\sigma : E_{\text{chall}} \to E_{\text{com}}$	$\xrightarrow{\sigma}$	Check that σ is an isogeny $E_{\text{chall}} \to E_{\text{com}}$

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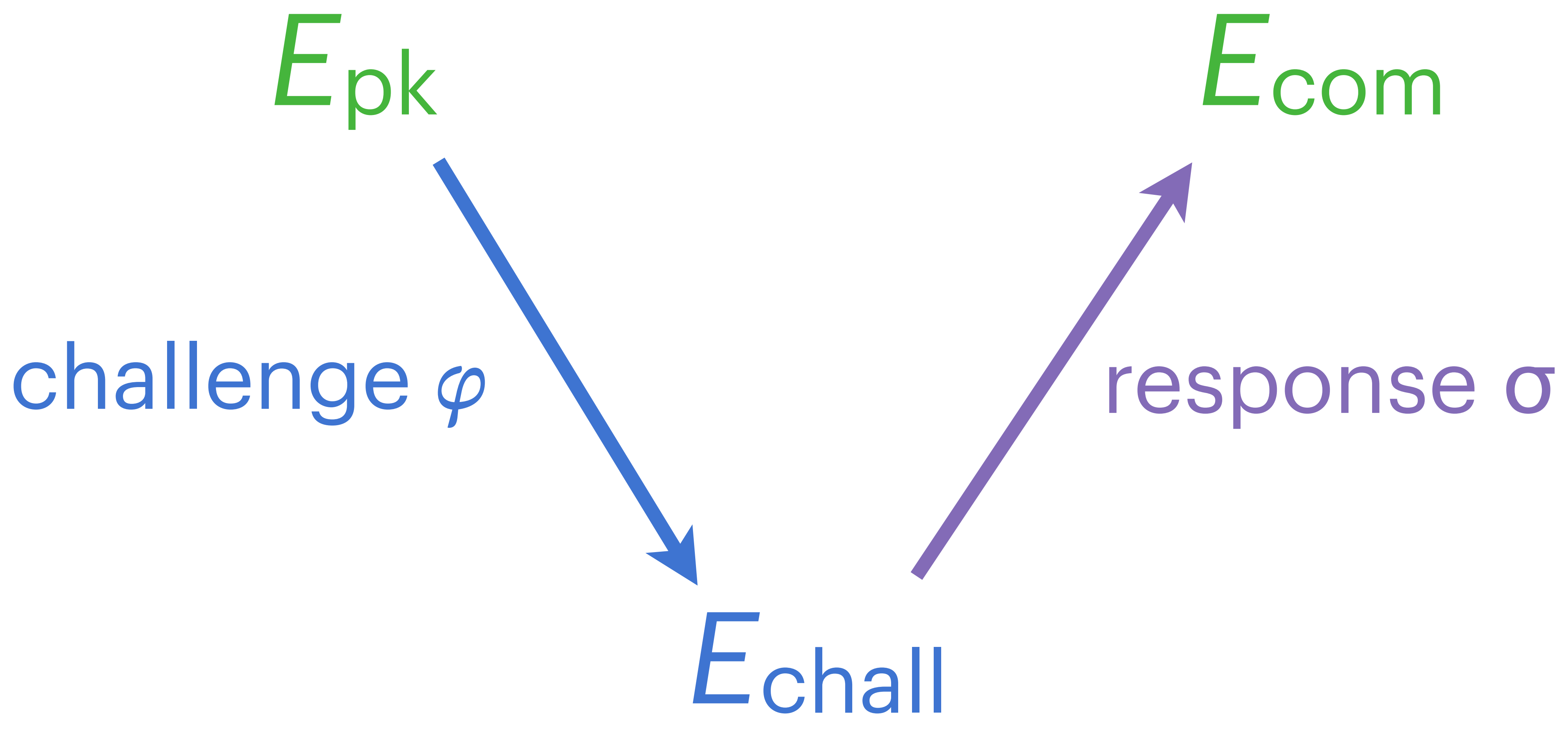

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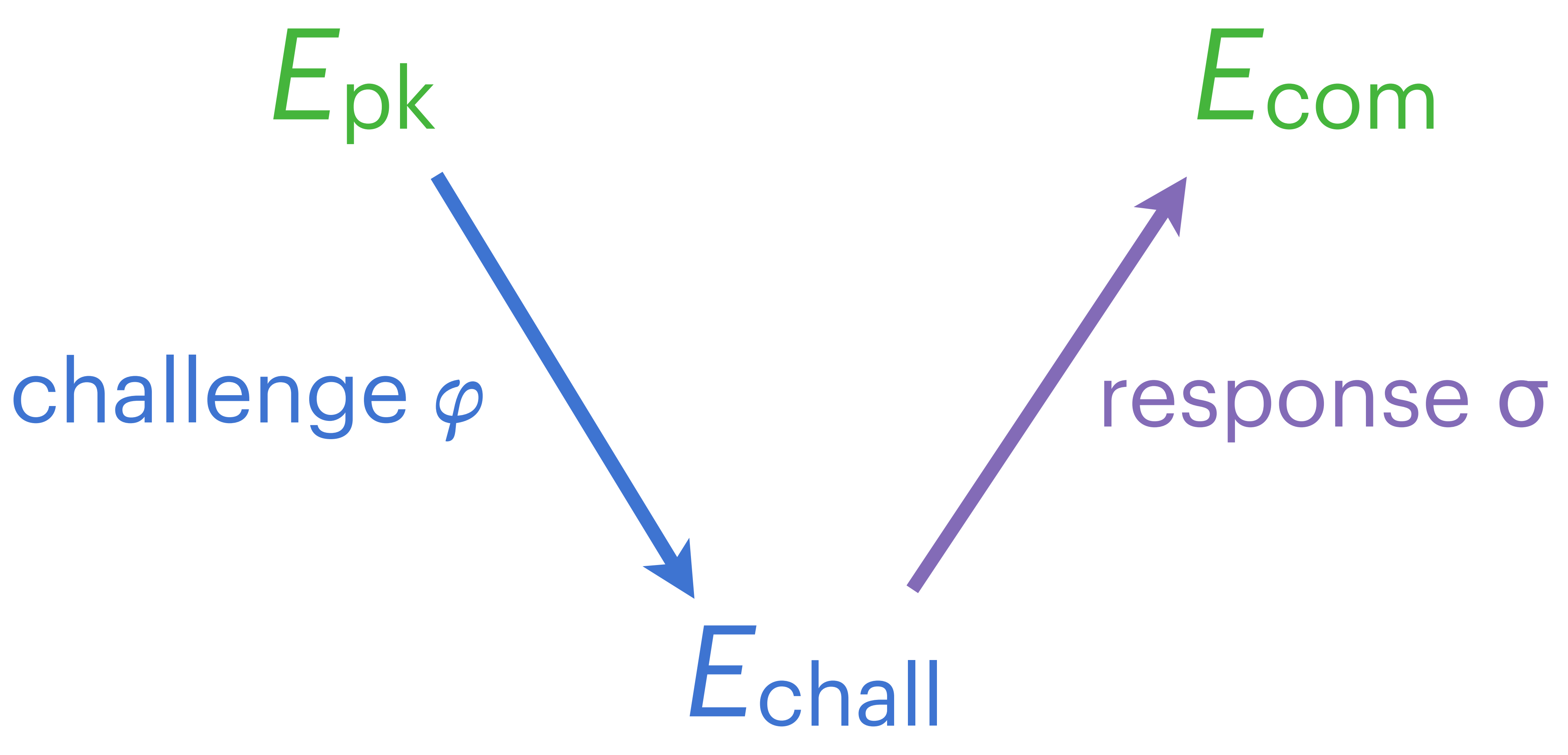

HV Zero-knowledge = **can generate transcripts with same distribution *without the secret key***

Honest transcript

E_{pk}

Honest transcript

1) Uniformly random curve

E_{pk} E_{com}

Honest transcript

1) Uniformly random curve

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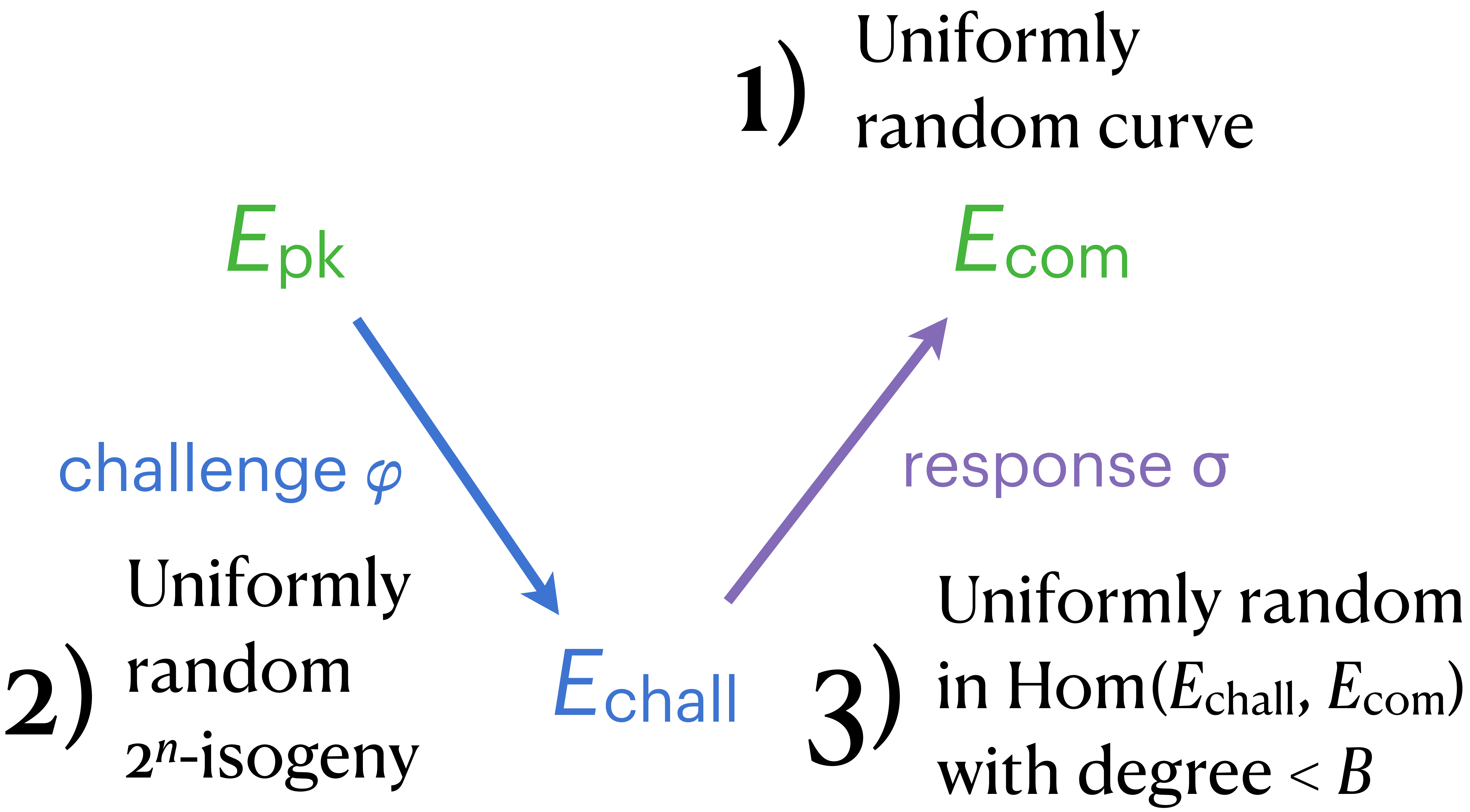
$$E_{\text{pk}} \xrightarrow{\text{challenge } \varphi} E_{\text{chall}}$$

2) Uniformly random 2^n-isogeny

Honest transcript



Simulated transcript

E_{pk}

Simulated transcript

$$E_{\mathsf{pk}} \xrightarrow{\text{challenge } \varphi} E_{\mathsf{chall}}$$

1) Uniformly random 2^n-isogeny

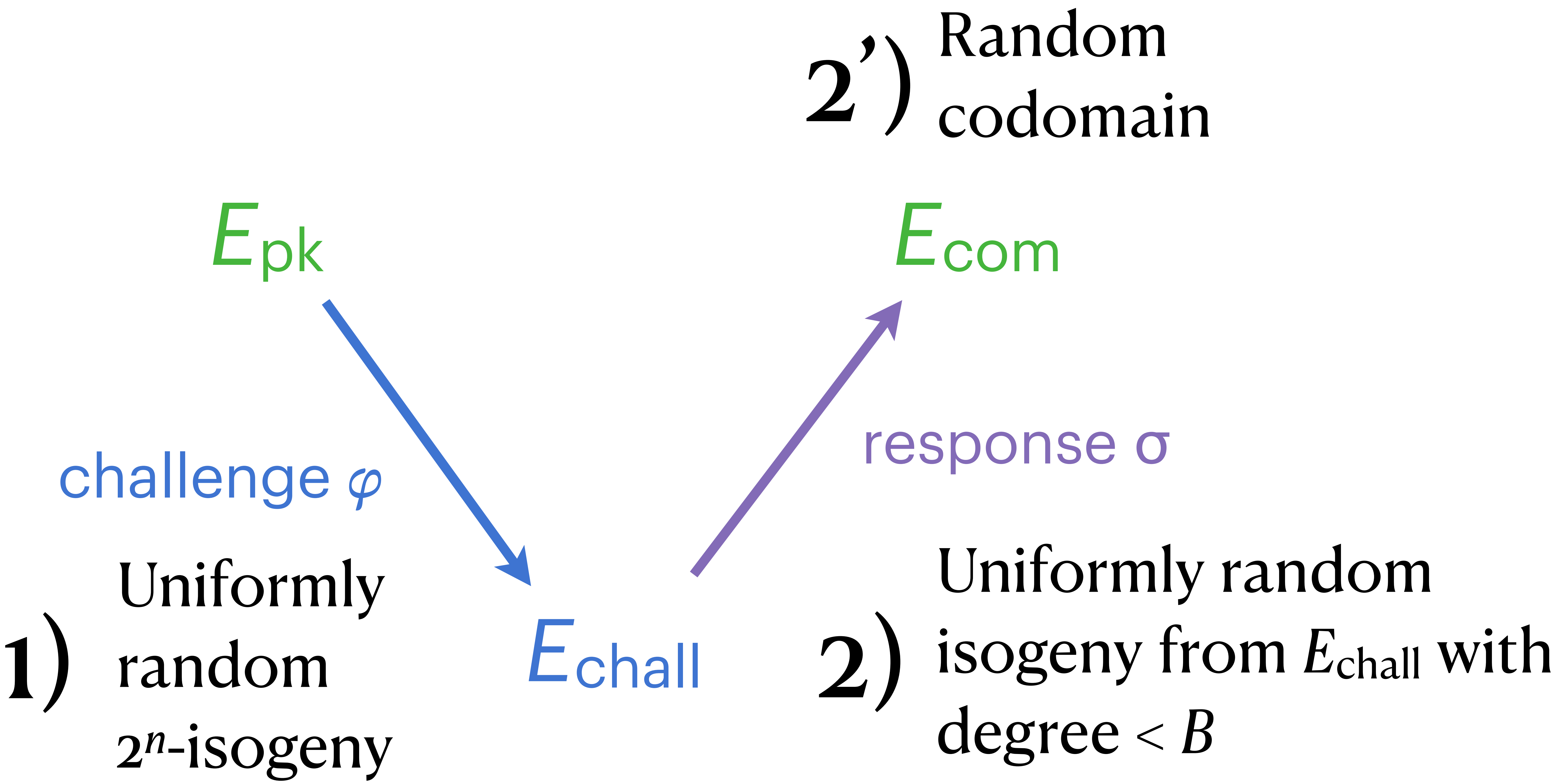
Simulated transcript

$$E_{\mathrm{pk}} \xrightarrow{\text{challenge } \varphi} E_{\mathrm{chall}} \xrightarrow{\text{response } \sigma} E_{\mathrm{com}}$$

1) Uniformly random 2^n-isogeny

2) Uniformly random isogeny from E_{chall} with degree $< B$

Simulated transcript



Simulated transcript

2’) Random codomain

E_{pk}

E_{com}

challenge φ

response σ

1) Uniformly random 2^n-isogeny

E_{chall}

2) Uniformly random isogeny from E_{chall} with degree $< B$

With B large enough, statistically indistinguishable from honest!

Simulated transcript

2’) Random codomain

E_{pk} $\xrightarrow{\text{challenge } \varphi}$ E_{chall} $\xrightarrow{\text{response } \sigma}$ E_{com}

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???

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RADIO

Random Any-Degree Isogeny Oracle:

- **Input:** an elliptic curve E, a bound $B > 0$
- **Output:** An efficient representation of uniformly distributed isogeny in

$$\{\varphi : E \to ? \mid \deg(\varphi) < B\}$$

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RADIO only extends that power to any degree. Powerful oracle?

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Assumption: EndRing still hard given a RADIO

Dimensions

2? 4? 8?

Picture by Beppe Rijs

How to verify?

$$\overset{\mathrm{End}(E_{chall})}{E_{chall}} \xrightarrow{\text{response } \sigma} \overset{\mathrm{End}(E_{com})}{E_{com}}$$

Response: σ represented as (d, P, Q, P', Q') where

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- $(P', Q') = (\sigma(P), \sigma(Q))$

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Force $2^{2n} - \deg(\sigma)$ to be a prime $\equiv 1 \bmod 4$

Two versions…

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SQIsign8D: no restriction on deg(σ) + degrees large enough for “stat. indisting.”

- **Provably** secure if **EndRing is hard given a RADIO**
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SQIsign4D: Force 2^n – deg(σ) to be a prime ≡ 1 mod 4 + use smaller degrees

- Security needs **heuristics**, but more compelling, simpler than original SQIsign
- NIST-I level (128 bits security): **64 bytes public key**, **109 bytes signature** (9x smaller than Falcon)
- Verification needs isogenies in dim 4; getting good [Dartois — eprint 2024/1180]
- Scales well to higher security

Dimension 2

SQIsign2D-West

Picture by Beppe Rijs

2D embedding of an isogeny

- Let $\sigma : E_{\mathrm{chall}} \to E_{\mathrm{com}}$ with $d = \deg(\sigma) < 2^{2n-2}$

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Use secret knowledge of $\text{End}(E_{\text{chall}})$...

Made fast with techniques from QFESTA [Nakagawa, Onuki - 2023] and Clapoti [Page, Robert - 2023]

SQIsign2D-West

[Basso, Dartois, De Feo, Leroux, Maino, Pope, Robert, W. — Asiacrypt 2024]

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- Verification: evaluation of an isogeny in dimension 2 (of degree 2^{2n}), much faster than all SQIsign predecessors
- Still very **compact**: NIST-I level **66 bytes public key**, **148 bytes signature**
- Security proof similar to most conservative predecessor (SQIsign8D)
  - **No heuristics**
  - Computational assumption: ***EndRing*** *is hard given access to an oracle that produces random isogenies of large degree (random-walks-on-steroids oracle)*
  - **Public key is uniformly distributed**: benefit from full force of worst-case to average-case reductions!

Performance

Performance in MCycles, for level of security NIST-I

Caution: non-uniform levels of optimizations... should only be indicative of an order of magnitude... timings in ms are extrapolated for ~3GHz

	Key gen.	Signing	Verif.
Original SQIsign	2800	4600	93
Optimized SQIsign	400	1880 620ms	29 10ms
SQIsignHD	190	115 38ms	?
SQIsign2D-West	60	160 53ms	9 3ms
SQIsign2D-West + heuristics	58	100 33ms	9

For NIST-V security level: cost x6