Computational foundations of

Isogeny-based cryptography

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October 2024, *25th Workshop on Elliptic Curve Cryptography*, Taipei, Taiwan

Isogenies

Elliptic curves, isogenies, isogeny graphs

Elliptic curves

- In crypto, we use elliptic curves over a **finite field**
- Elliptic curves are **groups**: you can add points together!

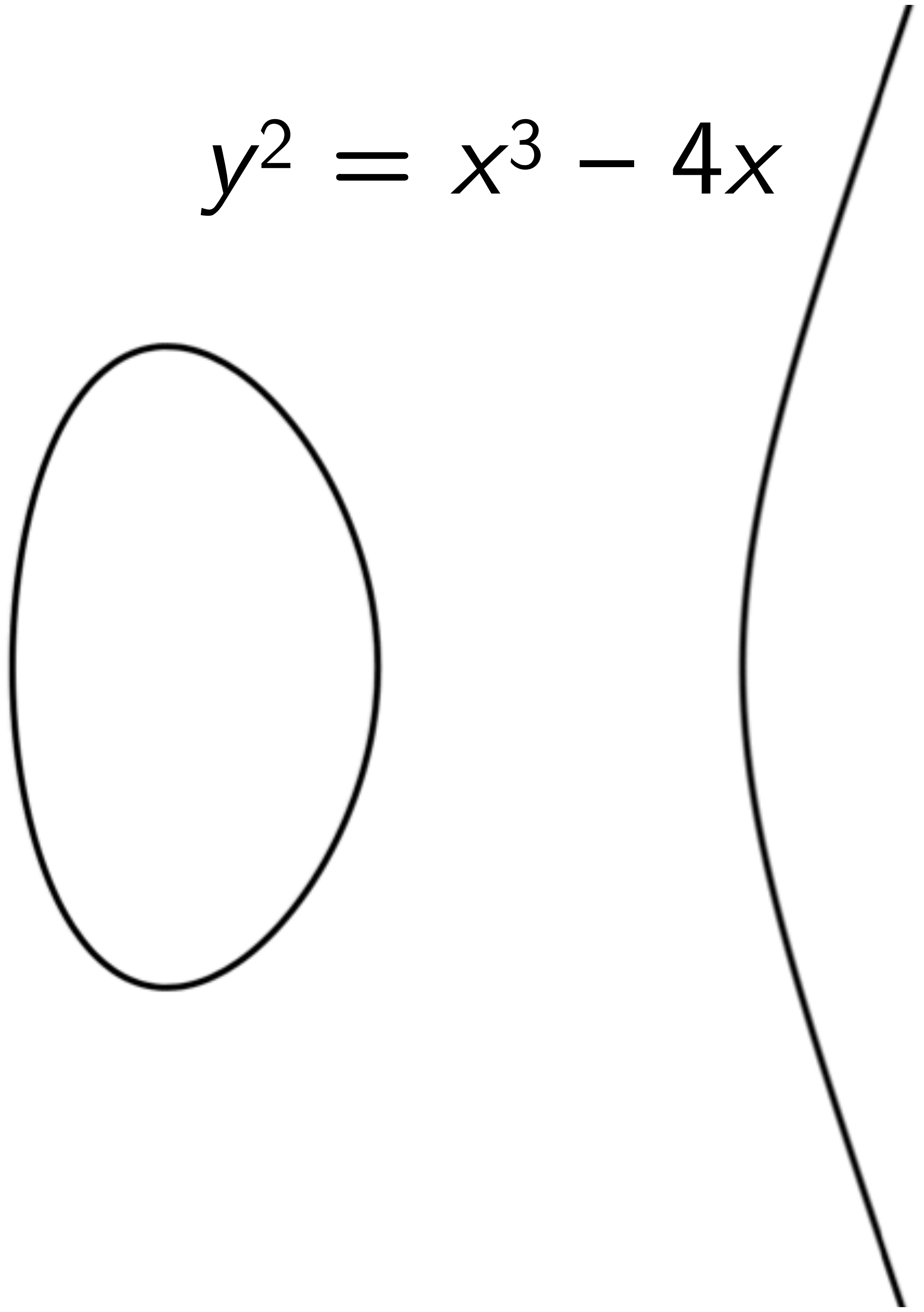
equations of the form

$$y^2 = x^3 + ax + b$$

$y^2 = x^3 + x$

E_1

$y^2 = x^3 - 4x$



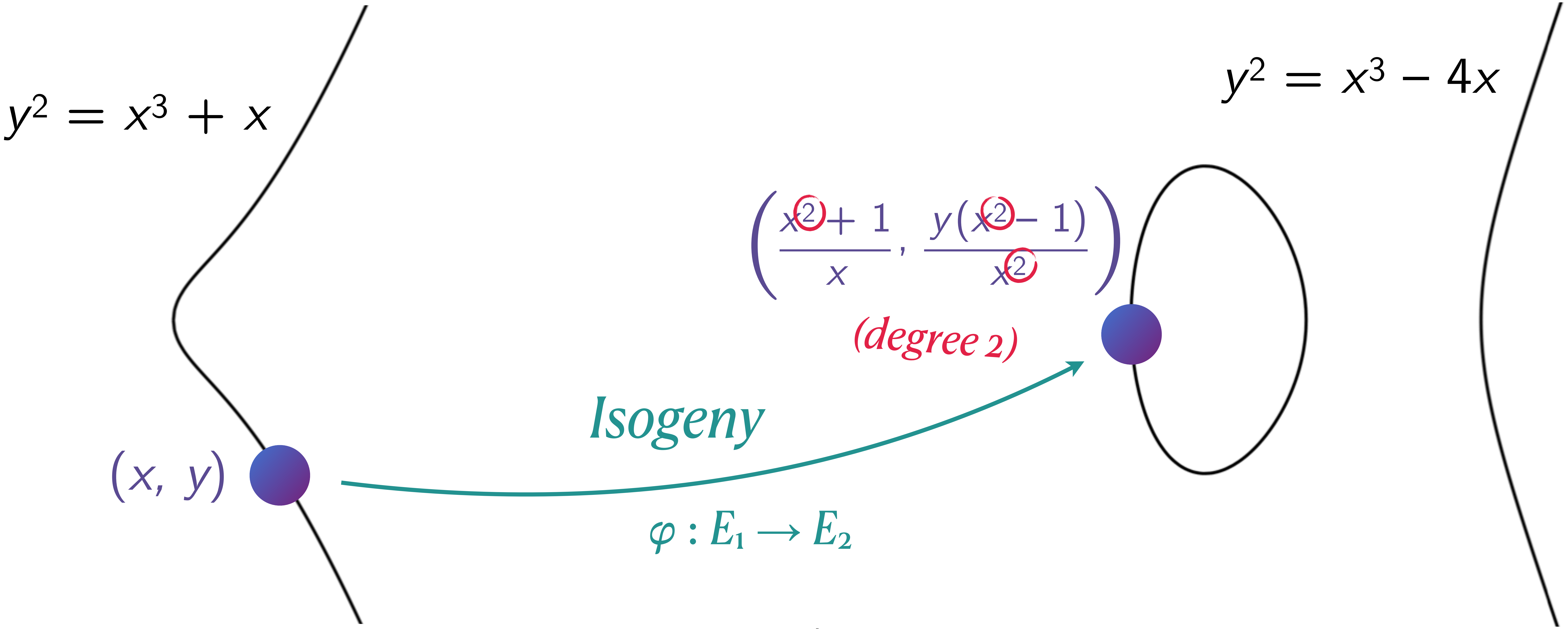
E_2

Elliptic curves

- In crypto, we use elliptic curves over a **finite field**
- Elliptic curves are **groups**: you can add points together!

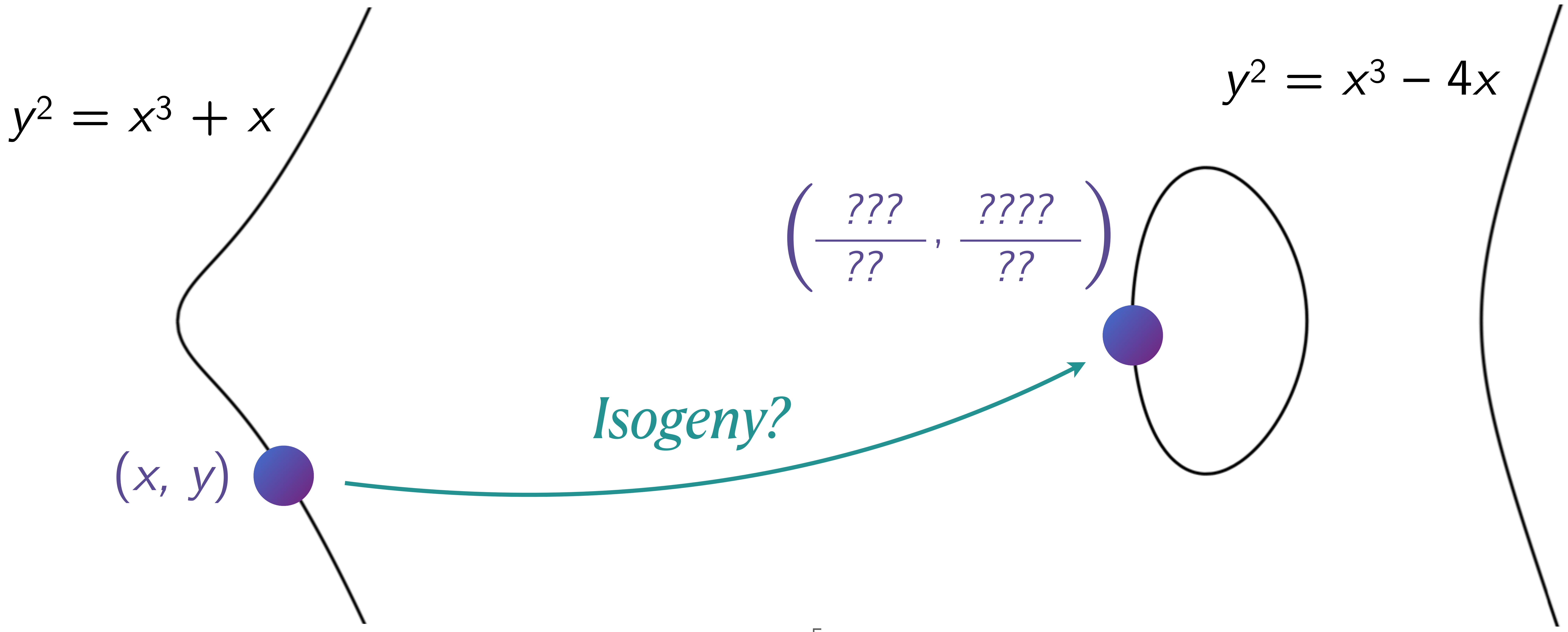
- Isogenies are group **homomorphisms**
- **Degree** = size of kernel

Sometimes, there is a formula to transform solutions from one equation to another



The *Isogeny* problem

Given E_1 and E_2 find an isogeny $\varphi : E_1 \to E_2$



The *Isogeny* problem

Given E_1 and E_2 find an isogeny $\varphi : E_1 \to E_2$

Hope: cryptosystems as secure as isogeny problem is hard

The isogeny problem = **Security of cryptosystems**

Hard even for quantum algorithms

Post-quantum cryptography

The *Isogeny* problem

Given E_1 and E_2 find an isogeny $\varphi : E_1 \to E_2$

solution typically has degree $\approx 2^{256}$

- The solution φ is an isogeny...
- How to represent an isogeny?

$$(x,\ y) \longmapsto \left(\frac{x^2+1}{x}, \frac{y(x^2-1)}{x^2}\right)$$

(degree 2)

fine for small degree...

- Build "big" isogenies as *formal compositions* of "small" ones

$$\deg(\varphi \circ \psi) = \deg(\varphi) \cdot \deg(\psi)$$

The *Isogeny* problem

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fine for small degree...

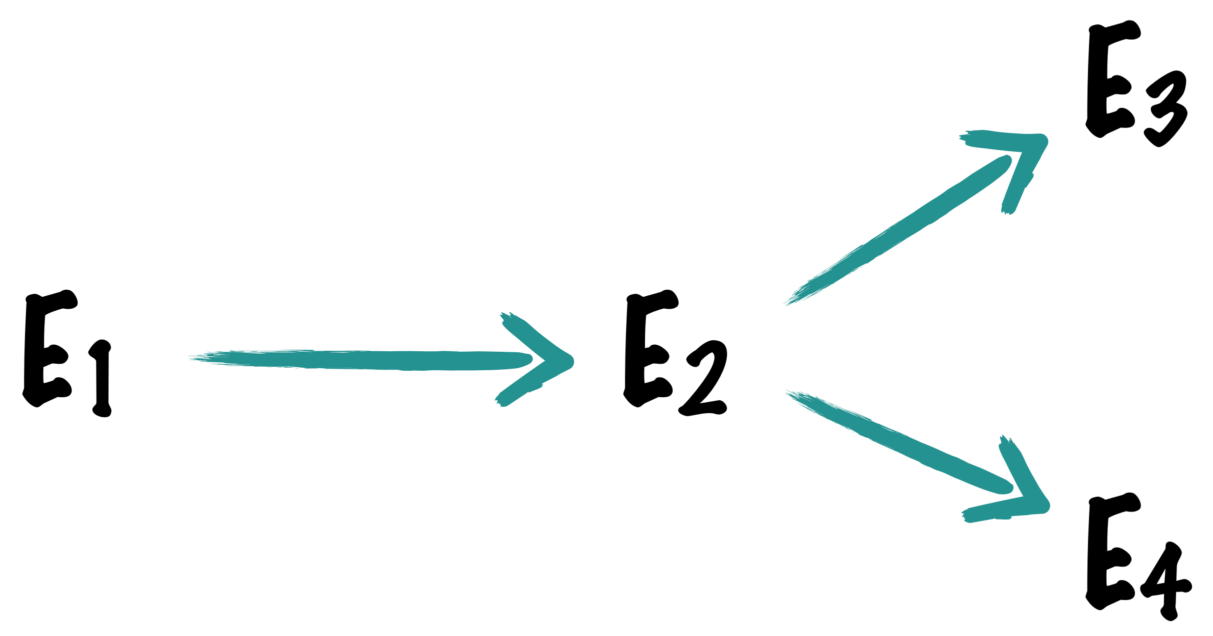
- Build "big" isogenies as *formal compositions* of "small" ones

$$E_1 \xrightarrow{2} E_2 \xrightarrow{2} E_3 \xrightarrow{2} \dots \xrightarrow{2} E_{257}$$

degree 2^{256}

Isogeny graph

- Fix small ℓ (say, $\ell = 2$). Can easily compute ℓ-isogenies



an isogeny of degree ℓ = an edge in a graph

Isogeny graph

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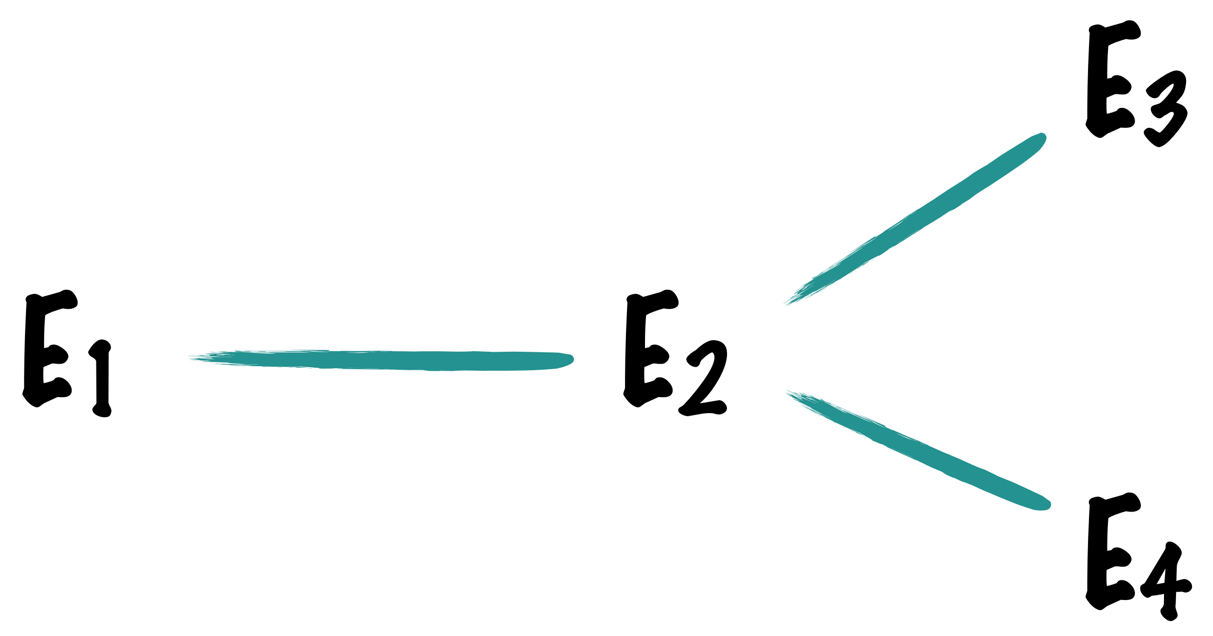
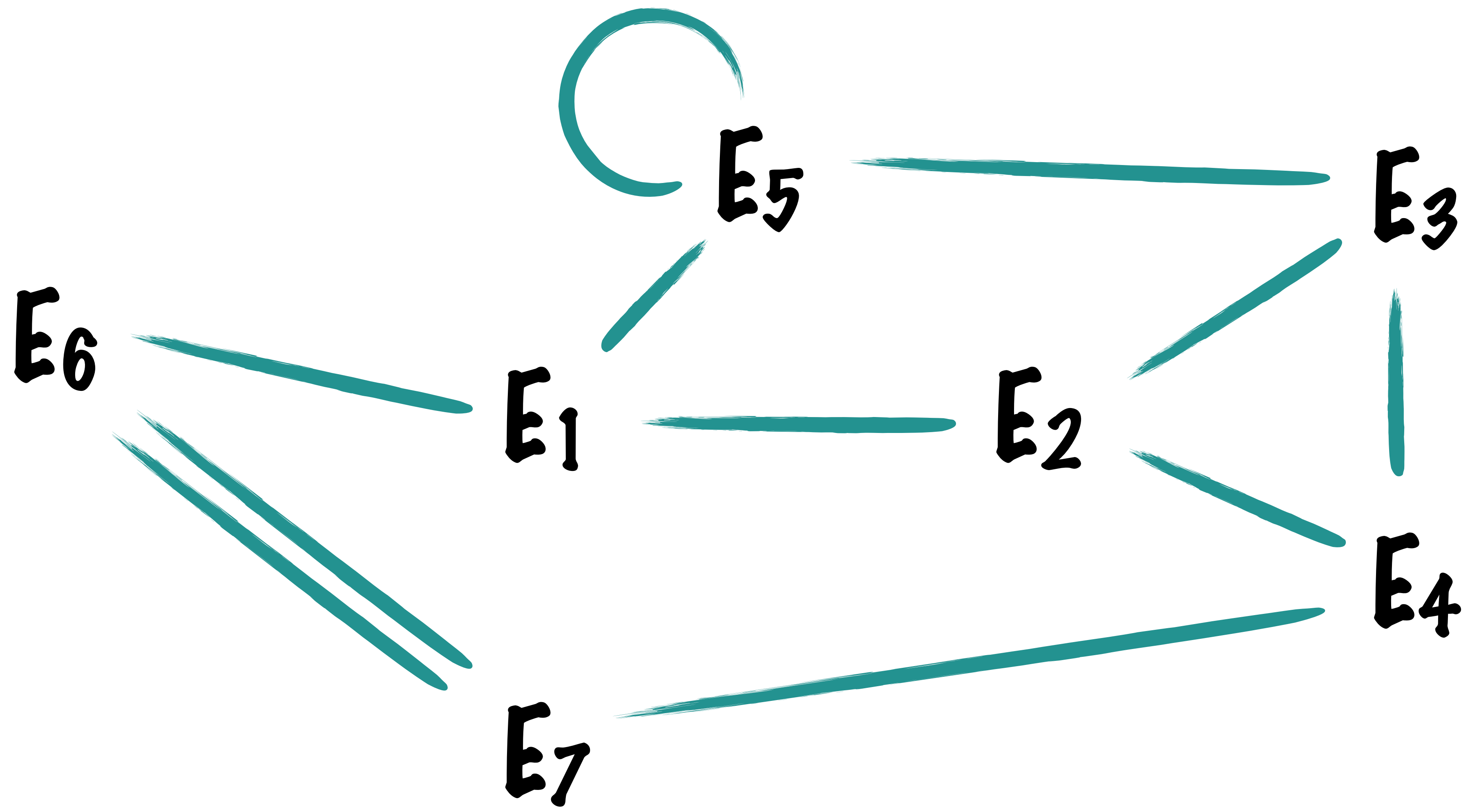


an isogeny of degree ℓ = an edge in a graph

$\exists\ \ell$-isogeny $E_1 \to E_2 \Rightarrow \exists\ \ell$-isogeny $E_2 \to E_1$

Isogeny graph

- Fix small ℓ (say, $\ell = 2$). Can easily compute ℓ-isogenies
- **The (supersingular...) ℓ-isogeny graph**



- $(\ell + 1)$-regular, **connected**, finite (all supersingular curves are defined over $\mathbb{F}_{p^2}$)

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178

The ℓ-isogeny path problem

The ℓ-*IsogenyPath* problem

Given E_1 and E_2 (supersingular) find an ℓ-isogeny path from E_1 to E_2

- Path finding in a graph of size ≈ $p/12$
- Hard for supersingular curves! Best known algorithm = generic graph algorithm
- Typical meaning of ***"the isogeny problem"***

Isogeny-based cryptography

Computational problems and cryptosystems

Isogeny-based cryptography

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The isogeny problem = **Security of cryptosystems**

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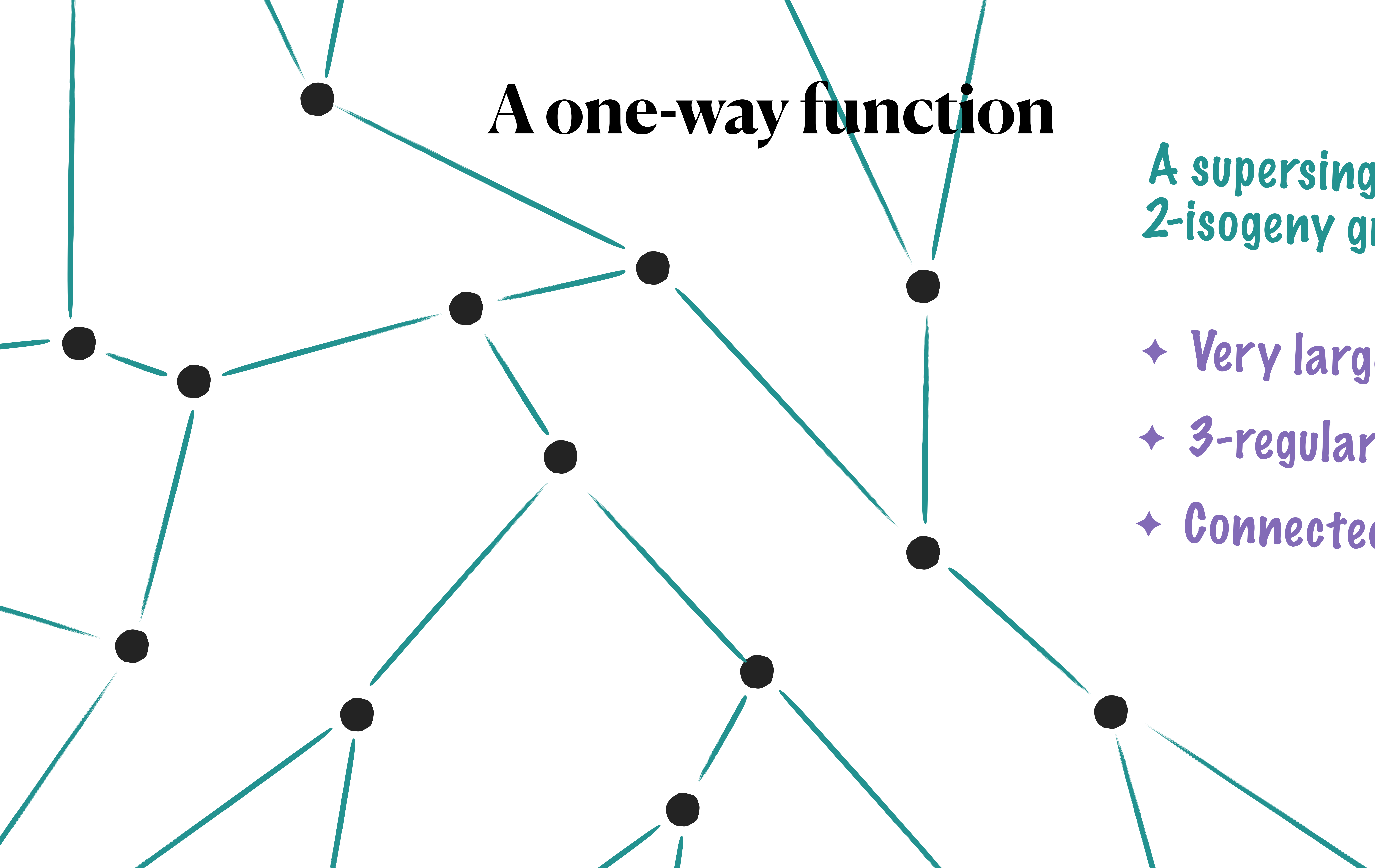
Post-quantum cryptography

A one-way function

- **One-way function:** a function $f : X \to Y$ which is
  - ➡ **Easy to evaluate:** given $x \in X$, it is easy to compute $f(x)$
  - ➡ **Hard to invert:** given $y \in Y$, it is hard to find some $x \in X$ such that $f(x) = y$

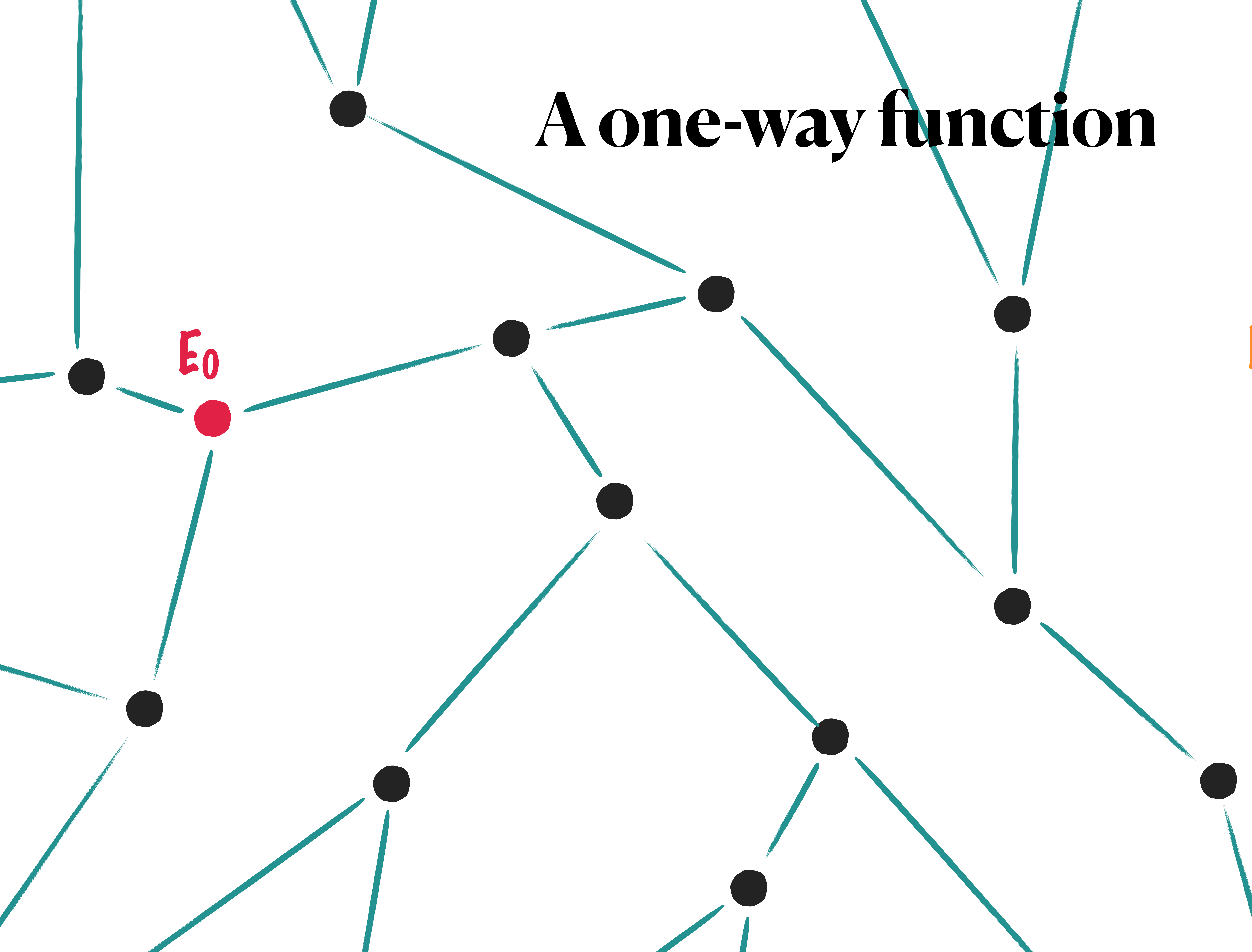
A one-way function



A supersingular 2-isogeny graph

- Very large
- 3-regular
- Connected

A one-way function

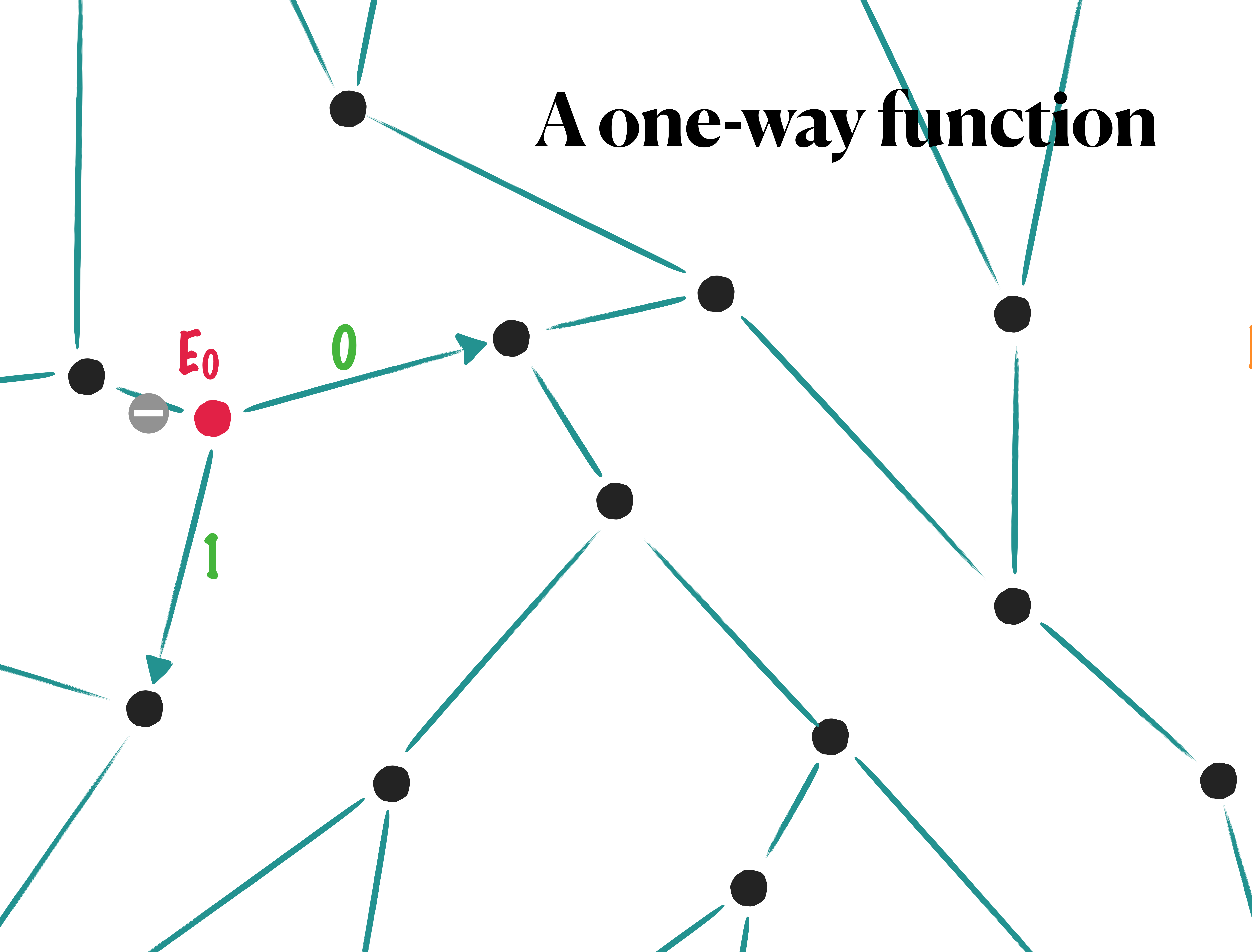


Fix a reference vertex E_0

Input: a message m in $\{0,1\}^*$

$m = 0001$

A one-way function



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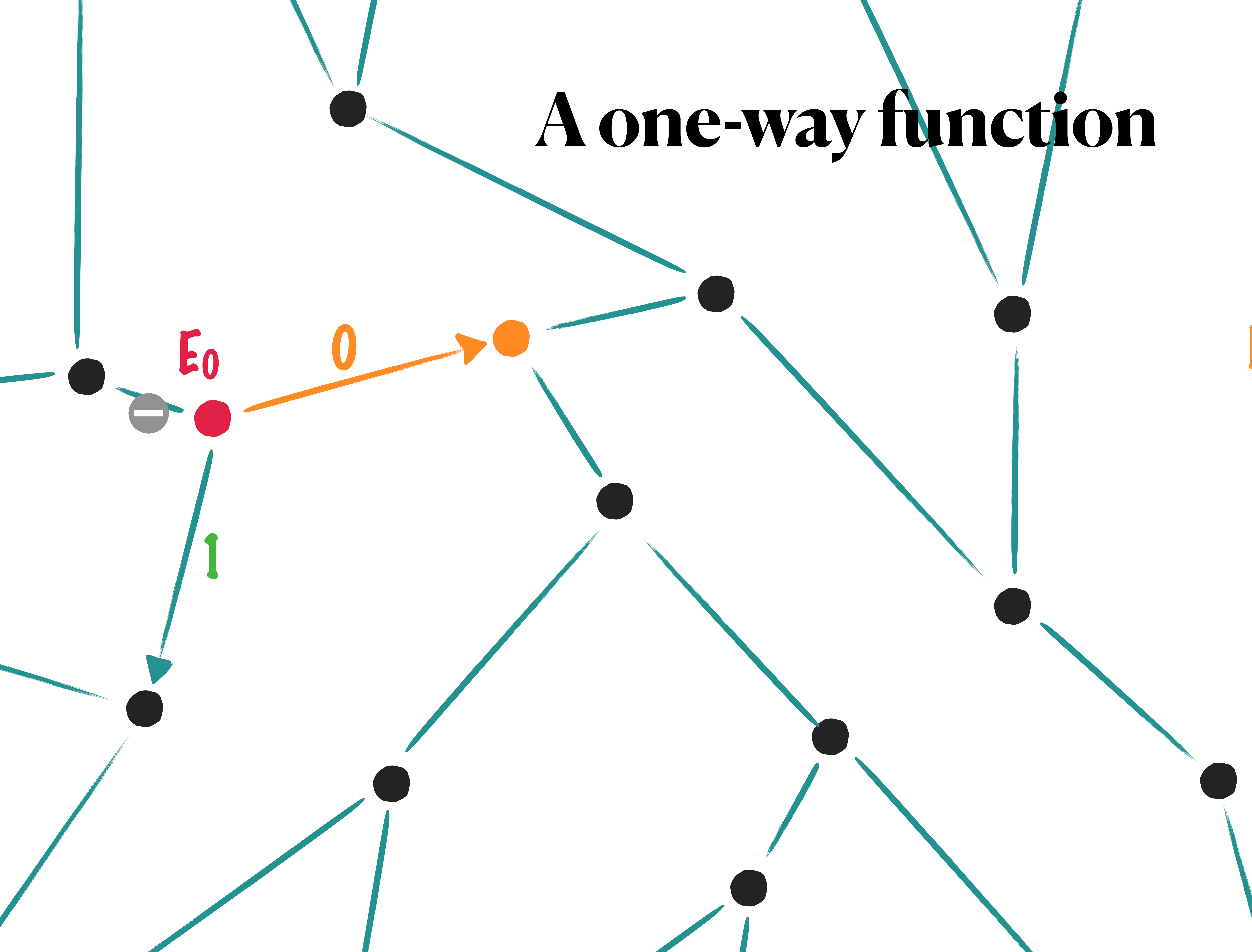

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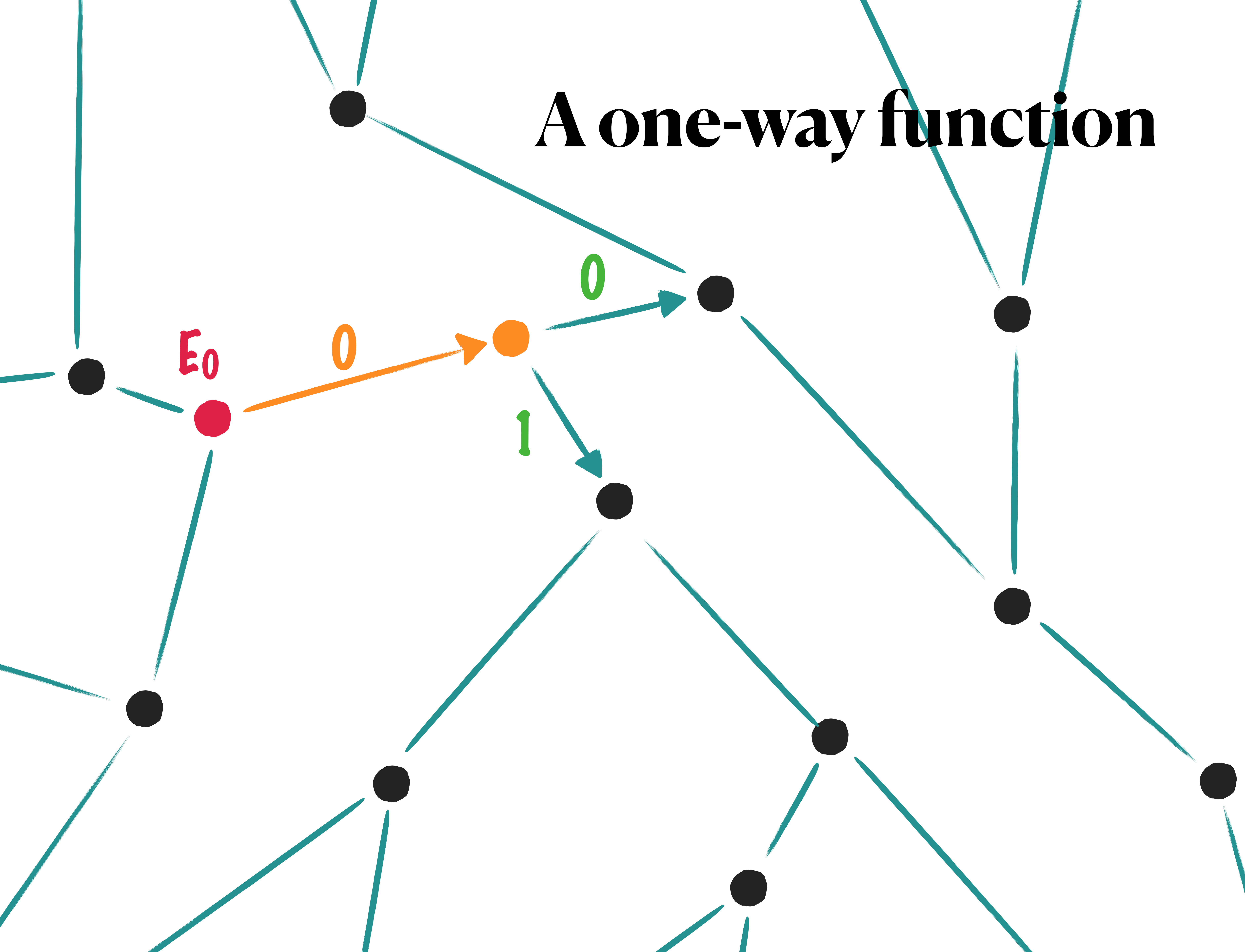

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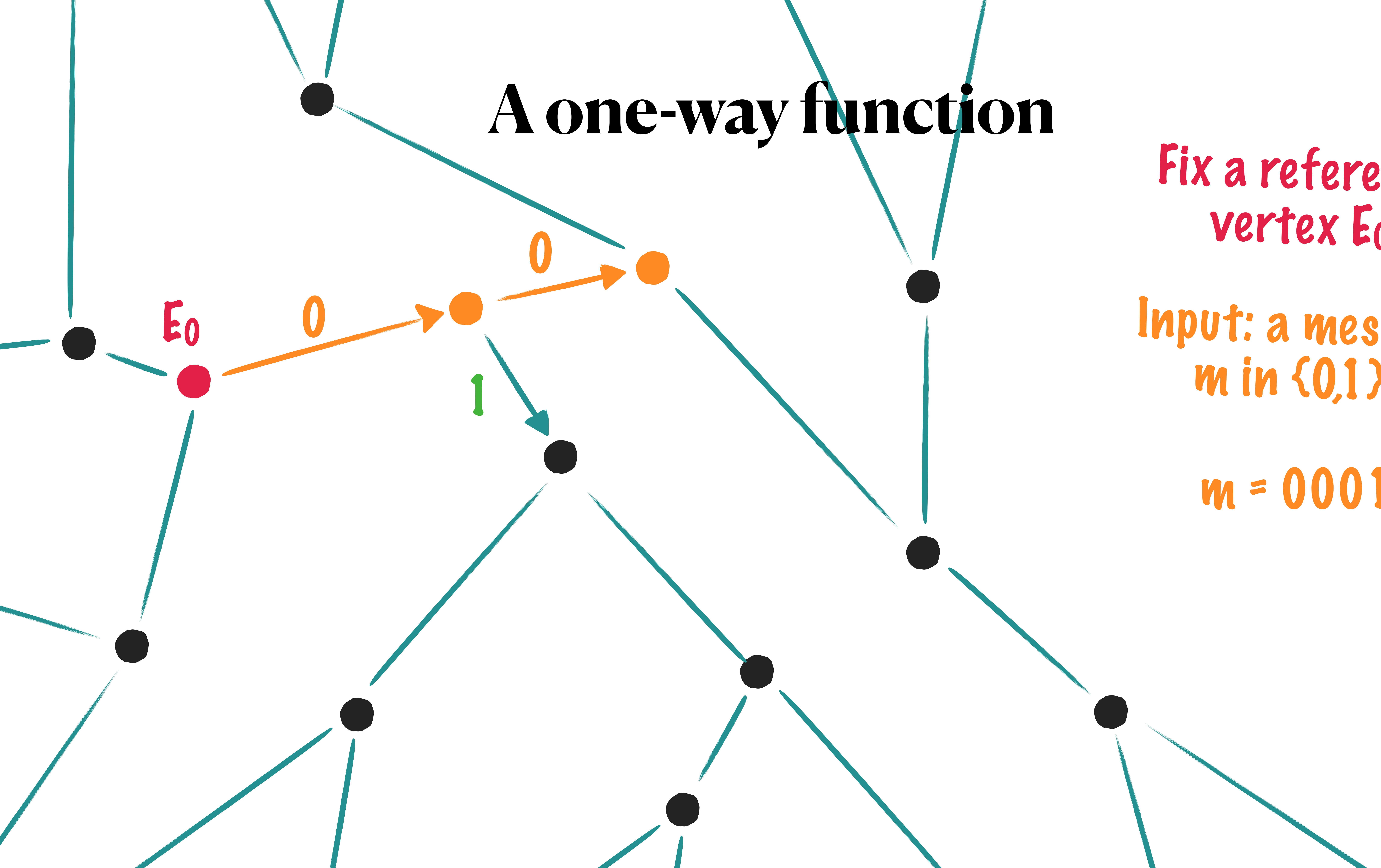

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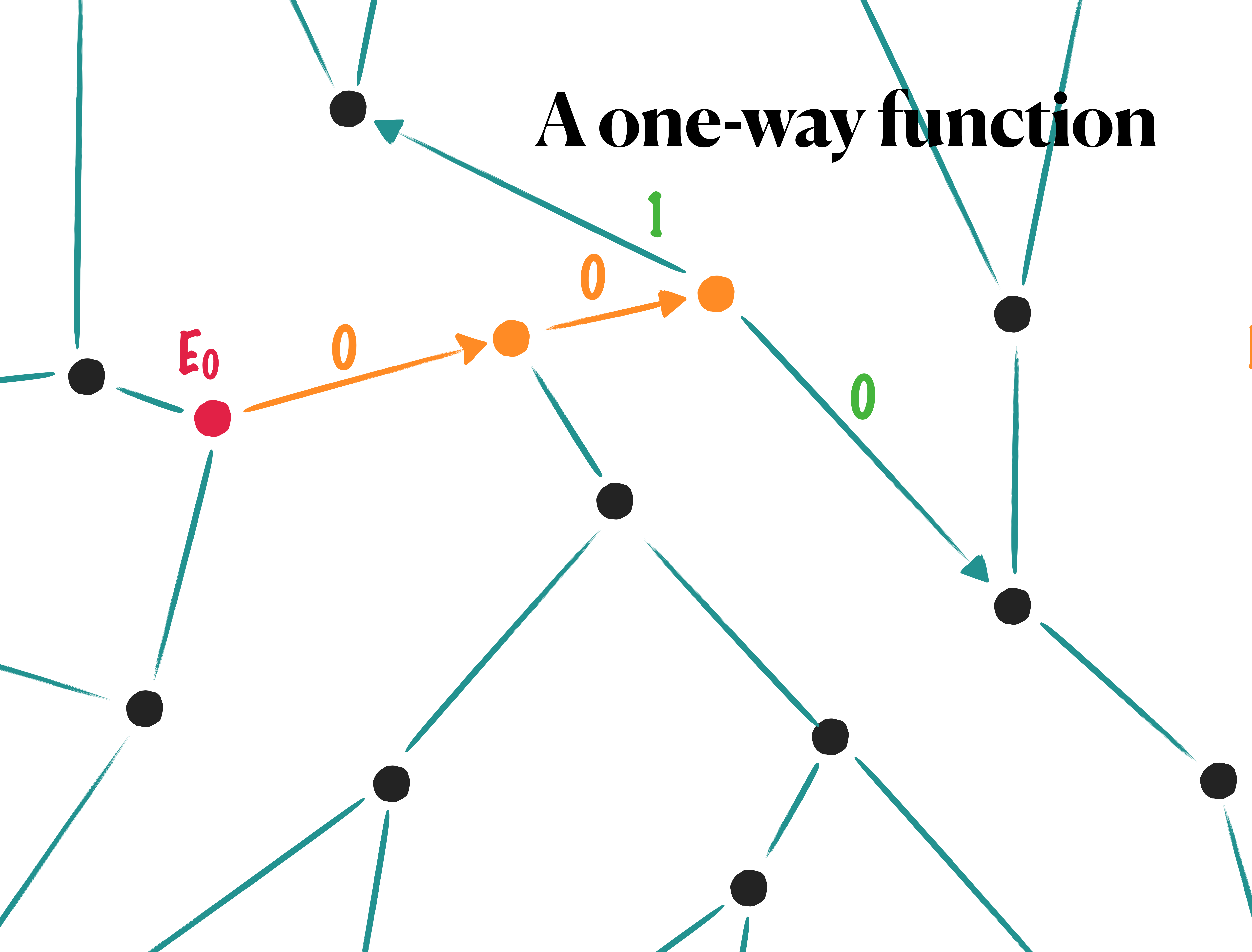

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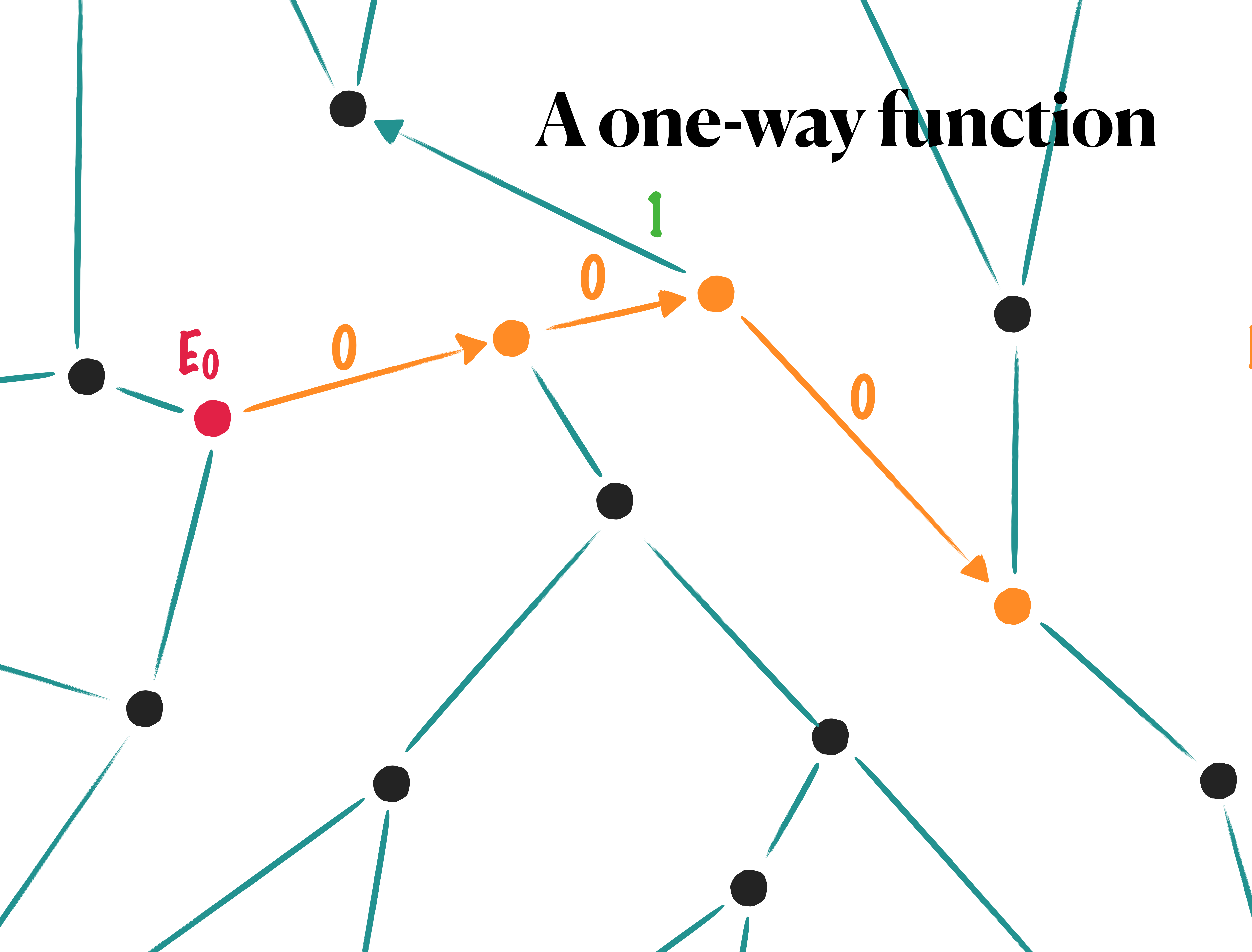

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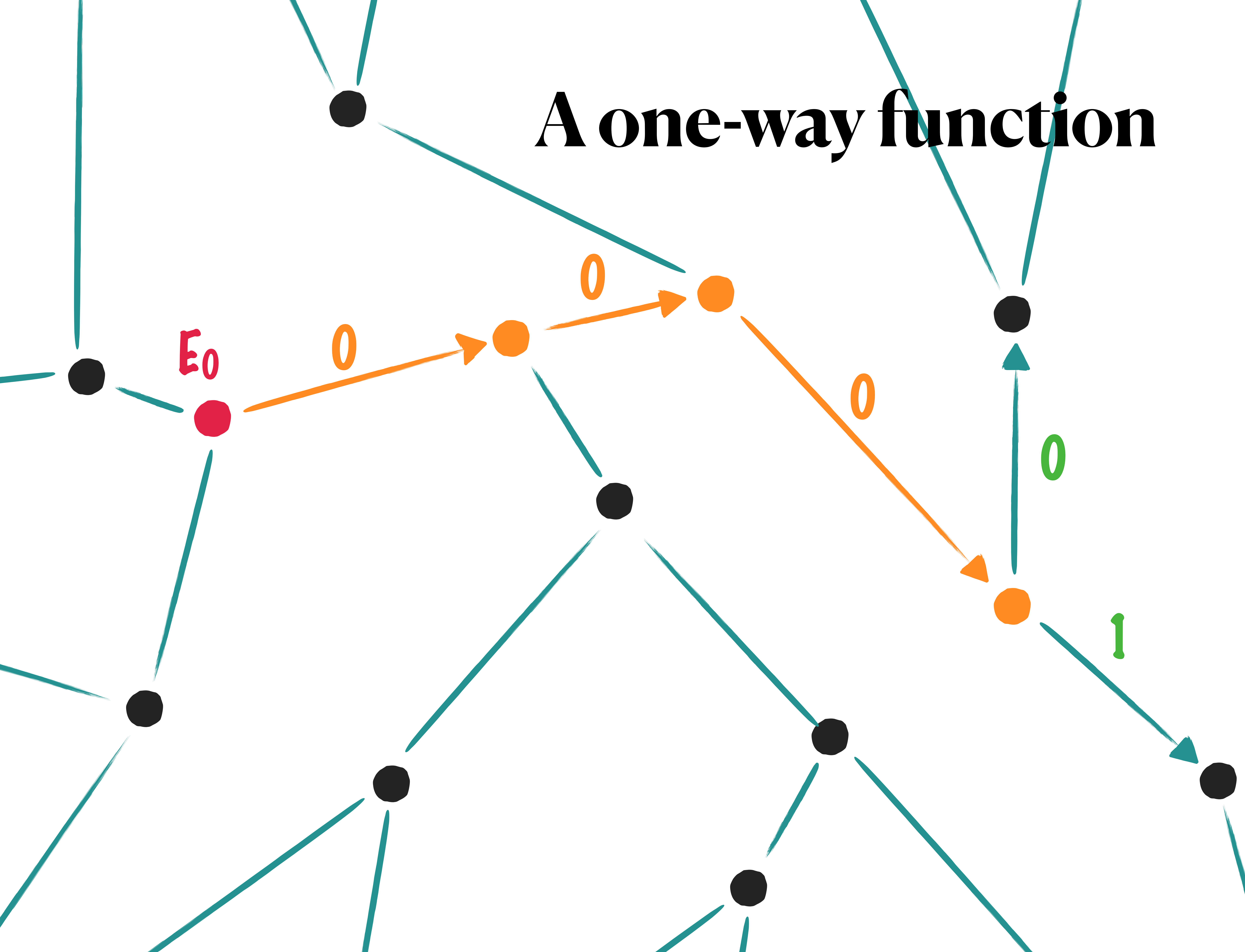

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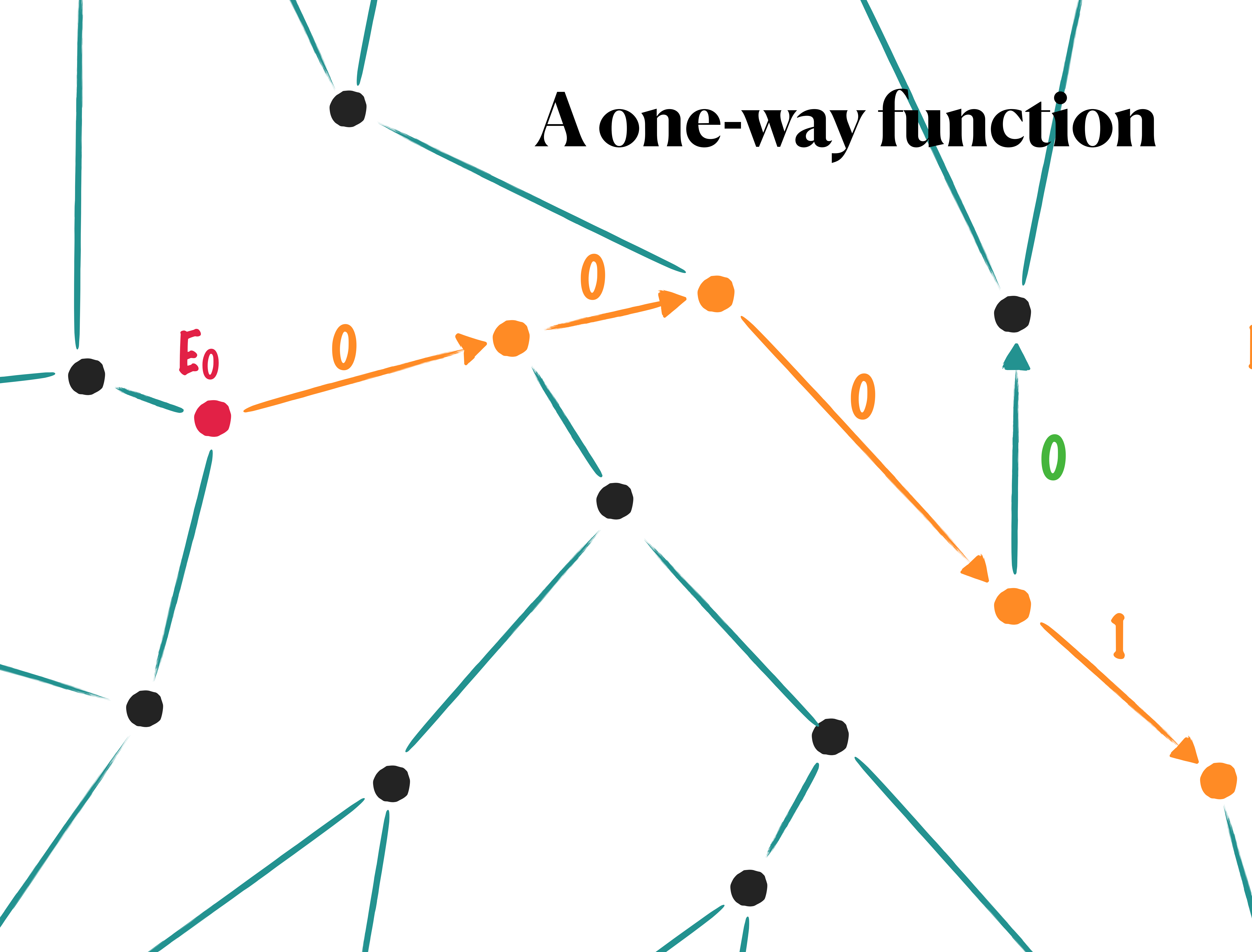

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Input: a message m in {0,1}*

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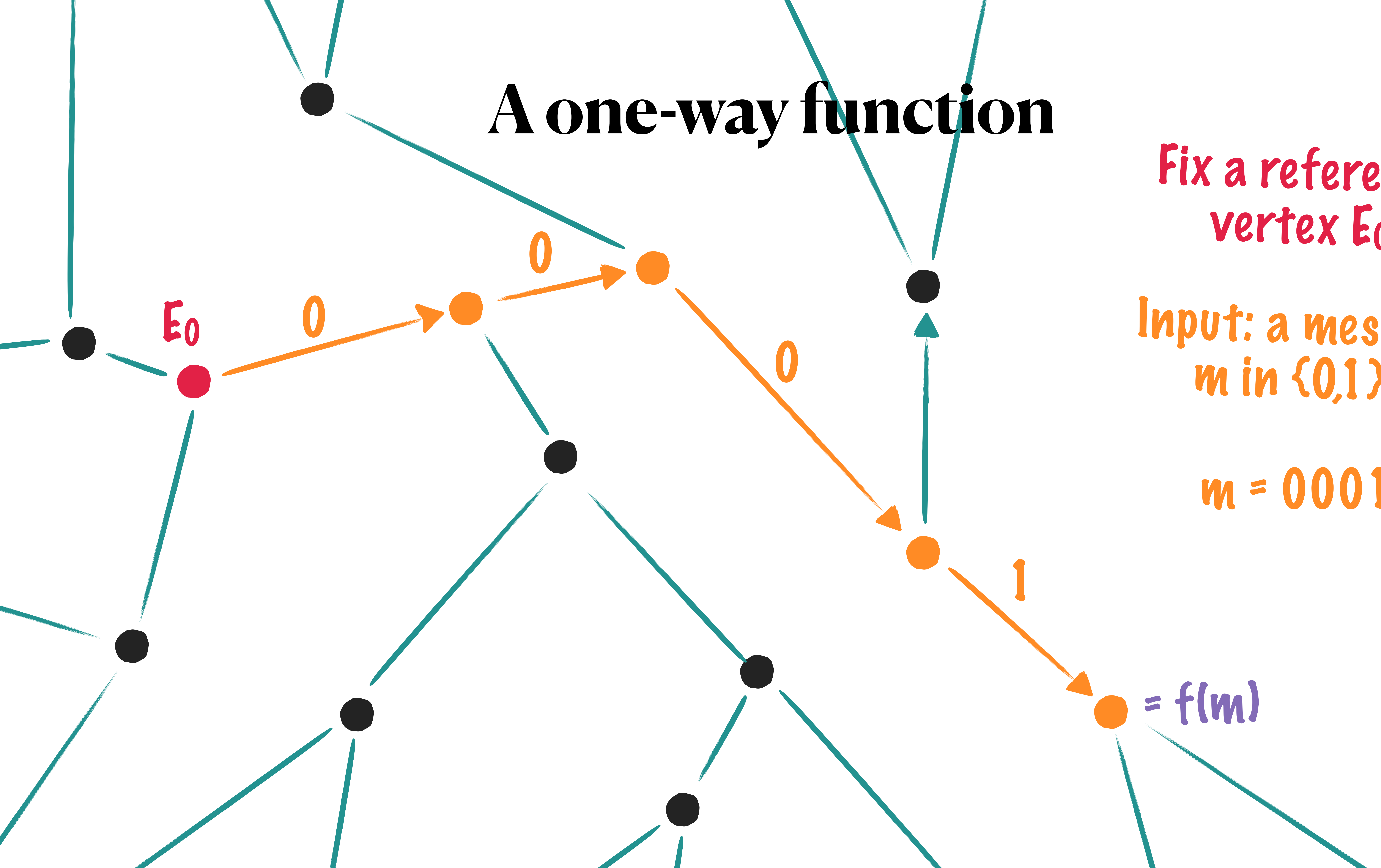

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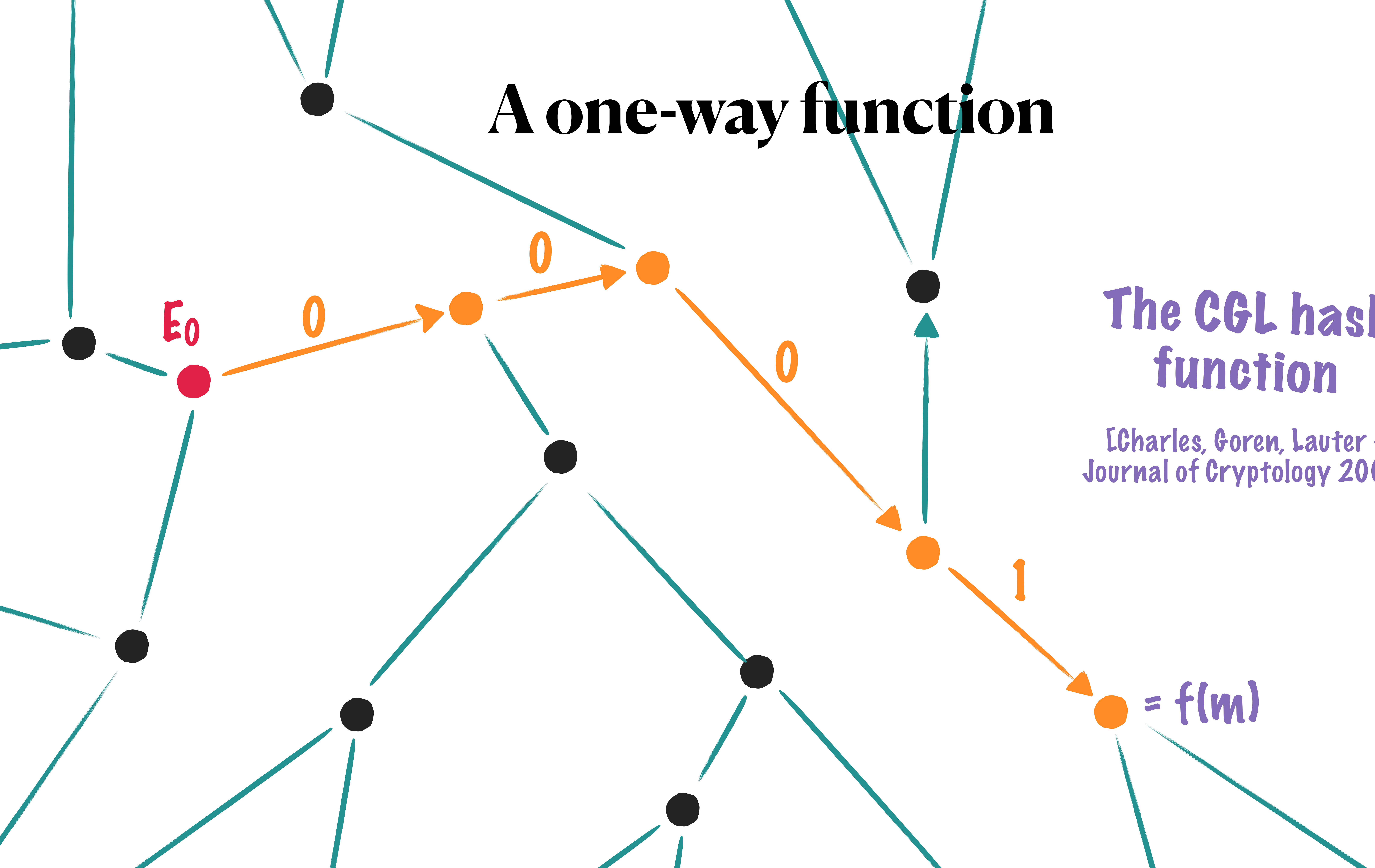
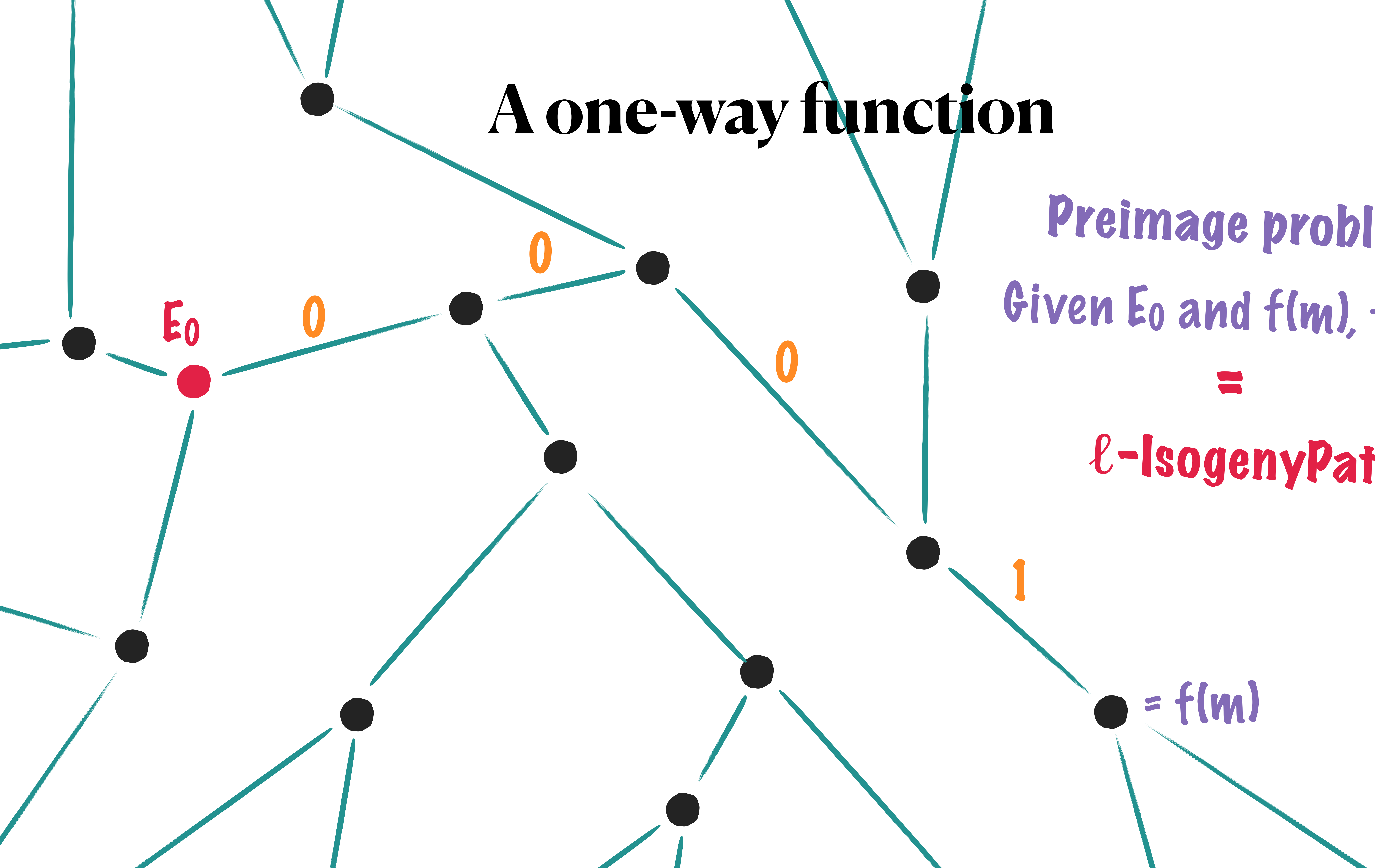


The CGL hash function

[Charles, Goren, Lauter - Journal of Cryptology 2009]

A one-way function



Preimage problem:

Given E_0 and $f(m)$, find m

=

ℓ-IsogenyPath

Isogeny-based cryptography

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Hard even for quantum algorithms

Post-quantum cryptography

A one-way function 👍

Isogeny-based cryptography

Reality: upper and lower bounds

?? $\leq$ **Security of cryptosystems** $\leq$ **ℓ-IsogenyPath**

ℓ-IsogenyPath	=	CGL hash function (preimage)
OneEnd	$\leq$	CGL hash function (collision)
OneEnd	$\leq$	SQIsign (soundness)
Vectorisation	$\leq$	CSIDH (key recovery)
~~**SSI-T**~~	~~$\leq$~~	~~SIDH (key recovery)~~

[Castryck, Decru]
[Maino, Martindale, Panny, Pope, W.]
[Robert]
Eurocrypt 2023

Endomorphisms

And the supersingular endomorphism ring problem

The CGL hash function

Is the CGL hash function collision-resistant?

Endomorphism ring

An **endomorphism** of E is an isogeny $\varphi : E \to E$ (or the zero map [0])

The **endomorphism ring** of E is End(E) = $\{\varphi : E \to E\}$

- $\varphi + \psi$ is pointwise addition: $(\varphi + \psi)(P) = \varphi(P) + \psi(P)$
- $\varphi\psi$ is the composition: $(\varphi\psi)(P) = \varphi(\psi(P))$

Multiplication by $m \in \mathbb{Z}$ is an endomorphism

$$[m] : E \to E : P \mapsto P + \ldots + P$$

It forms a subring $\mathbb{Z} \subset \mathrm{End}(E)$

Endomorphism ring

What is the structure of End(E)?

- It contains $\mathbb{Z} \subset \text{End}(E)$...
- (End(E), +) is a **lattice** of dimension 2 or 4

Endomorphism ring

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- (End(E), +) is a **lattice** of dimension 2 or 4

A curve E is **supersingular** if (End(E), +) is a lattice of dimension 4

Then, there is a $\mathbb{Z}$-basis 1, α_2, α_3, α_4: as a lattice,

$$\text{End}(E) = \mathbb{Z} \oplus \mathbb{Z}\alpha_2 \oplus \mathbb{Z}\alpha_3 \oplus \mathbb{Z}\alpha_4$$

Endomorphism ring

What is the structure of End(E)?

- It contains $\mathbb{Z} \subset \text{End}(E)$...
- (End(E), +) is a **lattice** of dimension 2 or 4
- Has a **Euclidean norm**: $||\alpha||^2 = \deg(\alpha)$
- **Scalar product** $\langle \alpha, \beta \rangle = (\deg(\alpha + \beta) - \deg(\alpha - \beta))/4$, volume...

A curve E is **supersingular** if (End(E), +) is a lattice of dimension 4

Then, there is a $\mathbb{Z}$-basis $1, \alpha_2, \alpha_3, \alpha_4$: as a lattice,

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The endomorphism ring problem

Given a supersingular E,
"compute End(E)"...

EndRing: Find four endomorphisms that form a basis of End(E)

Example

Example: $p \equiv 3 \pmod 4$, so $\mathbb{F}_{p^2} = \mathbb{F}_p(\alpha)$ where $\alpha^2 = -1$, and

Consider $E_0 : y^2 = x^3 + x$

Two non-trivial endomorphisms:

- $\pi : E_0 \to E_0 : (x, y) \longmapsto (x^p, y^p)$
- $\iota : E_0 \to E_0 : (x, y) \longmapsto (-x, \alpha y)$

$\pi^2 = [-p]$

$\iota^2 = [-1]$

and $\iota\pi = -\pi\iota$

$$\mathbf{End}(E_0) \overset{?}{=} \mathbb{Z} \oplus \mathbb{Z}\iota \oplus \mathbb{Z}\pi \oplus \mathbb{Z}\iota\pi$$

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$$\mathbf{End}(\boldsymbol{E_0}) = \mathbb{Z} \oplus \mathbb{Z}\iota \oplus \mathbb{Z}\frac{\iota + \pi}{2} \oplus \mathbb{Z}\frac{1 + \iota\pi}{2}$$

EndRing

The endomorphism ring problem

Given a supersingular E,
"compute End(E)"...

EndRing: Find four endomorphisms that form a basis of $\mathrm{End}(E)$

MaxOrder: Compute the "abstract structure" of $\mathrm{End}(E)$

- $\mathrm{End}(E)$ is isomorphic to a ring of quaternions. Find which!

Quaternion algebra

The **quaternion algebra $\boldsymbol{B_{p,\infty}}$** is the ring (for $p \equiv 3 \pmod 4$)

$$B_{p,\infty} = \mathbb{Q} \oplus \mathbb{Q}\, i \oplus \mathbb{Q}\, j \oplus \mathbb{Q}\, k$$

where $i^2 = -1$, $j^2 = -p$, and $k = ij = -ji$

End(E) is (isomorphic to) a discrete subrings of $B_{p,\infty}$

- End(E) is a **maximal order** in $B_{p,\infty}$
- There are **many** maximal orders in $B_{p,\infty}$

The endomorphism ring problem

Given a supersingular E,
"compute End(E)"...

EndRing: Find four endomorphisms that form a basis of End(E)

MaxOrder: Compute the "abstract structure" of End(E)

- Find a subring of $\boldsymbol{B_{p,\infty}}$ isomorphic to End(E)

Example

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Consider $E_0 : y^2 = x^3 + x$

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$$\mathbf{End}(E_0) = \mathbb{Z} \oplus \mathbb{Z}\iota \oplus \mathbb{Z}\frac{\iota + \pi}{2} \oplus \mathbb{Z}\frac{1 + \iota\pi}{2}$$

EndRing

$$\simeq \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}\frac{i + j}{2} \oplus \mathbb{Z}\frac{1 + ij}{2} \subset B_{p,\infty}$$

MaxOrder

The CGL hash function

The *OneEnd* problem

Given E (supersingular) find ***one*** endomorphism $\alpha \in \mathrm{End}(E) \setminus \mathbb{Z}$

The endomorphism ring problem

Given a supersingular E,
"compute End(E)"...

EndRing: Find four endomorphisms that form a basis of End(E)

MaxOrder: Compute the "abstract structure" of End(E)

- Find a subring of $\boldsymbol{B_{p,\infty}}$ isomorphic to End(E)

OneEnd: Find a single non-scalar endomorphism in $\alpha \in \mathrm{End}(E) \setminus \mathbb{Z}$

Foundations

Relations between problems

Which is hardest? Easiest?

EndRing

ℓ-IsogenyPath

OneEnd

MaxOrder

Isogeny

Relating OneEnd to EndRing

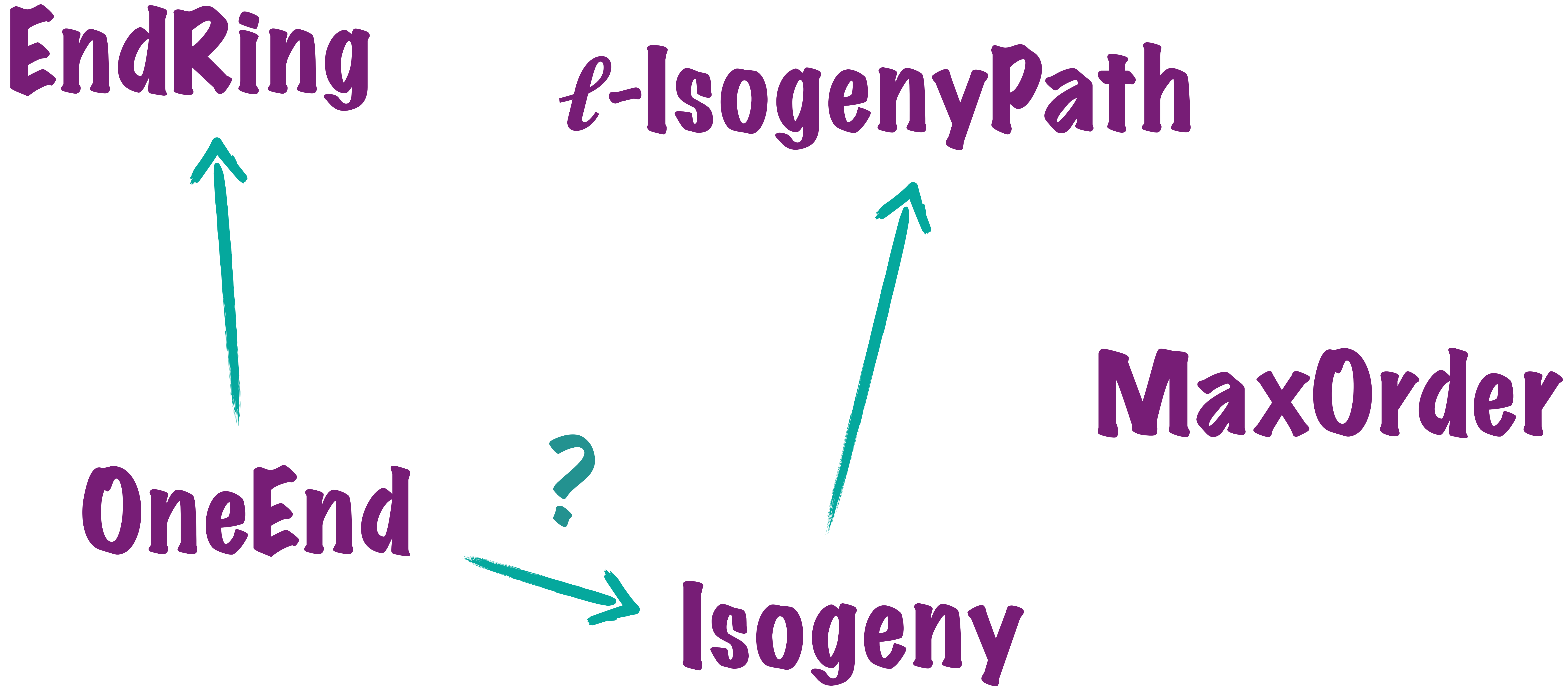
Suppose we can solve **EndRing**. Can we solve **OneEnd**?

Given E, we solve **OneEnd** for E as follows:

1. Solve **EndRing** for E, finding a basis $1, \alpha_2, \alpha_3, \alpha_4$ of $\mathrm{End}(E)$
2. Return α_2

We have that $\alpha_2 \in \mathrm{End}(E) \setminus \mathbb{Z}$ because α_2 is not in $\mathrm{span}(1) = \mathbb{Z}$

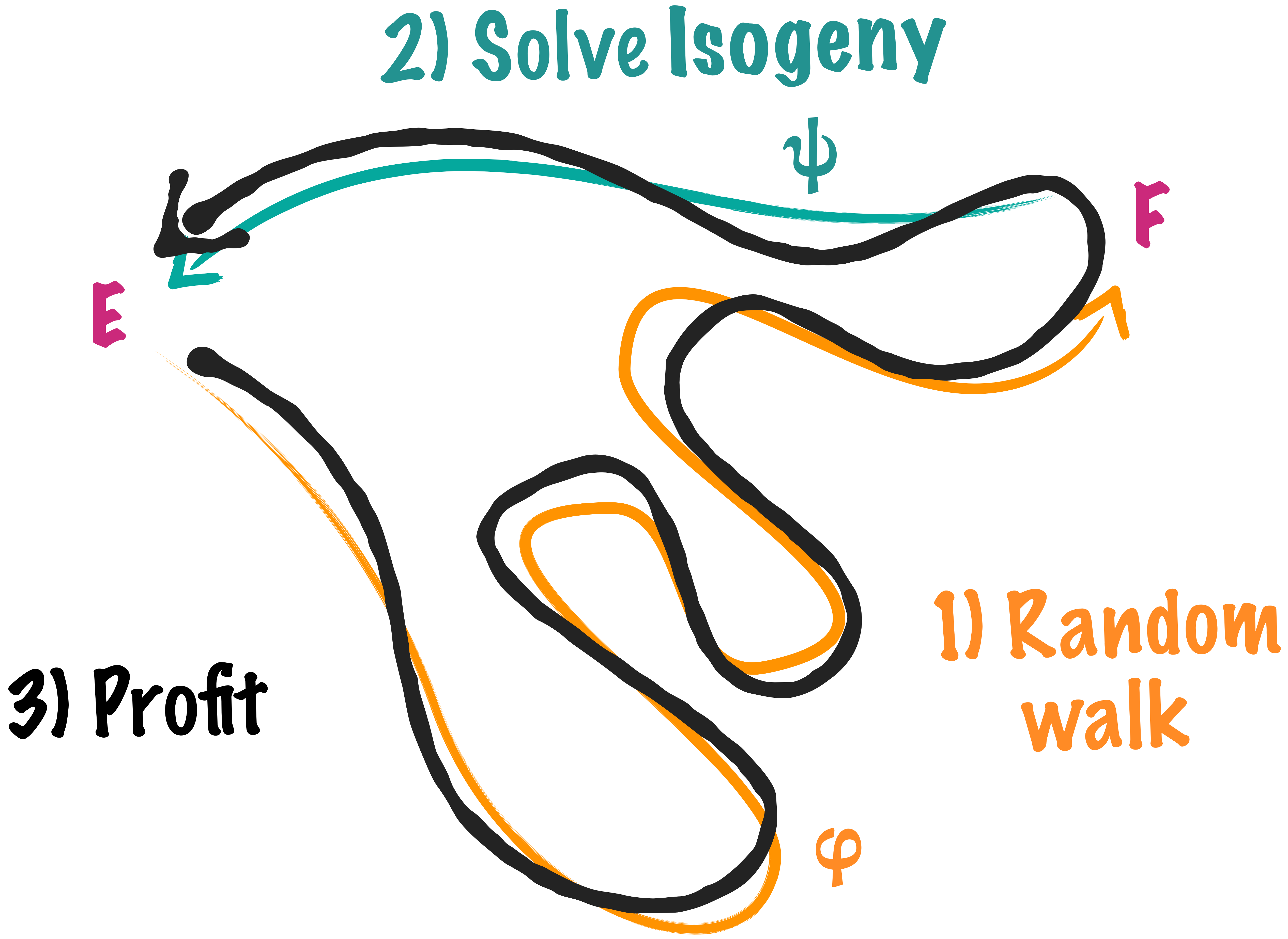
Which is hardest? Easiest?



Relating OneEnd to Isogeny

Suppose we can solve **Isogeny**.
Can we solve **OneEnd**?

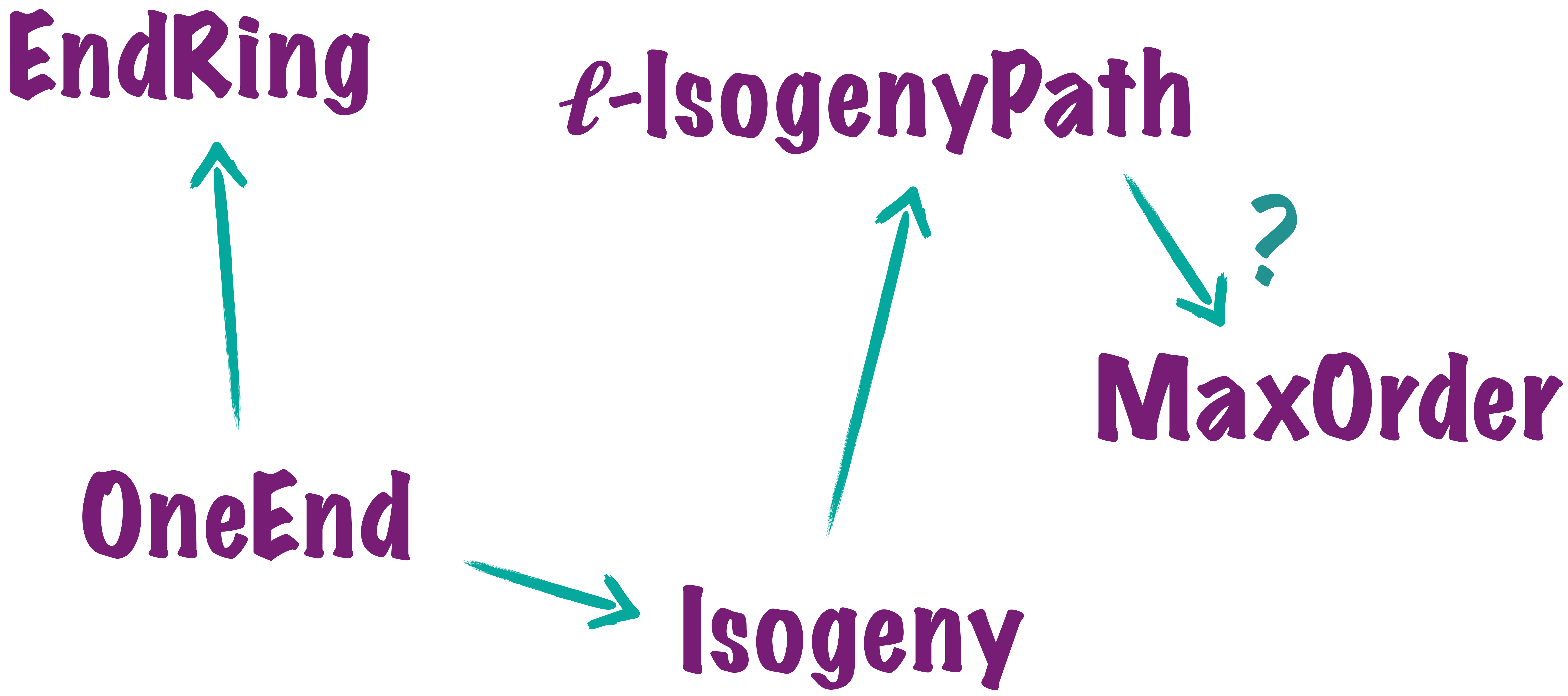
How to find endomorphisms of E:



Does $\psi \circ \varphi \in \mathbb{Z}$?

- Not if φ is long enough, and has cyclic kernel

Which is hardest? Easiest?



Isogeny World | Quaternion World

Deuring correspondence

Isogeny World	Quaternion World
Supersingular curves $\boldsymbol{E}$ over $\mathbb{F}_{p^2}$ (up to isomorphism)	Maximal orders $\mathcal{O}$ in $B_{p,\infty}$ $\mathcal{O} \simeq \mathbf{End}(\boldsymbol{E})$ (up to isomorphism)
Isogenies $\varphi : \boldsymbol{E} \to \boldsymbol{E'}$	$(\mathcal{O},\mathcal{O}')$-ideals $\boldsymbol{I}$, $\mathcal{O} \simeq \mathbf{End}(\boldsymbol{E})$ and $\mathcal{O}' \simeq \mathbf{End}(\boldsymbol{E'})$

HARD | **HARD? EASY?**

MaxOrder

ℓ-Isogeny Path:

Given E and E',
find φ : E $\longrightarrow$ E' of degree ℓ^n

ℓ-Quaternion Path:

Given $\mathcal{O}$ and $\mathcal{O}'$, find an
$(\mathcal{O},\mathcal{O}')$-ideal I of norm ℓ^n

Solving the Quaternion Path Problem

Theorem: There exists an algorithm that **solves the ℓ-quaternion path problem** in expected polynomial time (assuming GRH).

Full proof under GRH: **[W. – FOCS 2021]** *The supersingular isogeny path and endomorphism ring problems are equivalent*.

Much faster, but heuristic algorithm: **[Kohel, Lauter, Petit, Tignol – ANTS 2014]** *On the quaternion ℓ -isogeny path problem*.

The "KLPT" algorithm

Isogeny World

Quaternion World

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HARD → **MaxOrder** → **EASY**

ℓ-Isogeny Path:

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ℓ-Quaternion Path:

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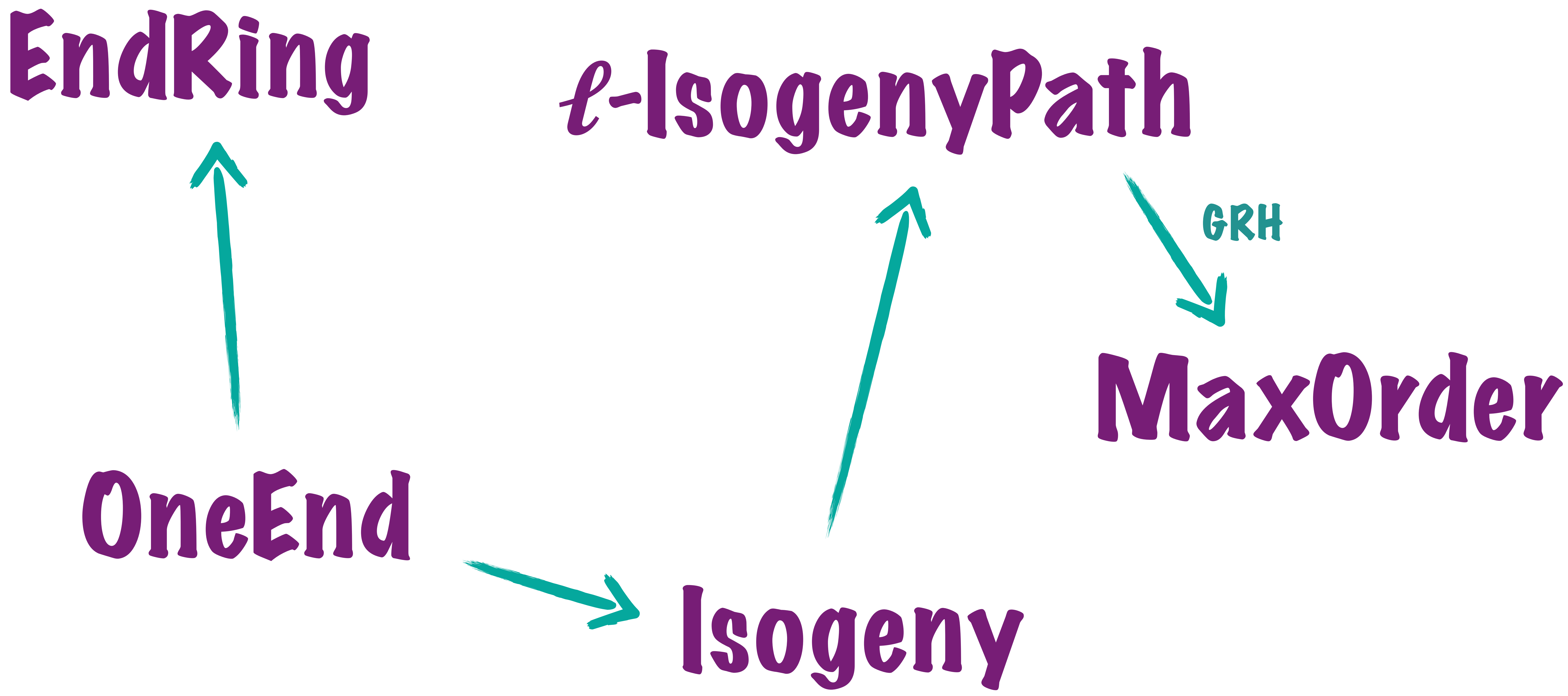
Which is hardest? Easiest?



Which is hardest? Easiest?

$$\text{EndRing} \overset{\text{GRH}}{\longleftrightarrow} \text{MaxOrder} \overset{\text{GRH}}{\longleftrightarrow} \ell\text{-IsogenyPath}$$

Proof assuming GRH:

[W. – FOCS 2021] *The supersingular isogeny path and endomorphism ring problems are equivalent.*

Earlier heuristic reductions:

[Petit, Lauter – preprint 2017] *Hard and Easy Problems for Supersingular Isogeny Graphs.*

[Eisenträger, Hallgren, Lauter, Morrison, Petit – Eurocrypt 2018] *Supersingular isogeny graphs and endomorphism rings: Reductions and solutions.*

Which is hardest? Easiest?

$$\begin{tikzcd} \text{EndRing} \arrow[r, leftrightarrow, "\text{GRH}"] & \text{MaxOrder} \arrow[r, leftrightarrow, "\text{GRH}"] & \ell\text{-IsogenyPath} \\ \text{OneEnd} \arrow[u] \arrow[r] & \begin{array}{l}\text{CGL collision-resistance}\\ \text{SQIsign soundness}\end{array} \arrow[ul] \end{tikzcd}$$

Which is hardest? Easiest?

EndRing $\overset{\text{GRH}}{\longleftrightarrow}$ MaxOrder $\overset{\text{GRH}}{\longleftrightarrow}$ ℓ-IsogenyPath

$\uparrow\downarrow$

[Page, W. – Eurocrypt 2024] *The supersingular Endomorphism Ring and One Endomorphism problems are equivalent.*

OneEnd $\longleftrightarrow$ CGL collision-resistance
SQIsign soundness

Which is hardest? Easiest?

$$
\begin{tikzcd}
\textsf{EndRing} \arrow[r, leftrightarrow, "\text{GRH}"] \arrow[d, leftrightarrow] \arrow[dr, phantom, "\text{Unconditional!}"] & \textsf{MaxOrder} \arrow[r, leftrightarrow, "\text{GRH}"] & \ell\text{-}\textsf{IsogenyPath} \\
\textsf{OneEnd} \arrow[r] & \textsf{Isogeny} \arrow[ur] &
\end{tikzcd}
$$

Which is hardest? Easiest?

$$\begin{tikzcd}
\text{EndRing} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \text{MaxOrder} \arrow[d, leftrightarrow] & \ell\text{-IsogenyPath} \arrow[l, "\text{GRH}"'] \\
\text{OneEnd} \arrow[r, leftrightarrow] & \text{Isogeny} \arrow[ur]
\end{tikzcd}$$

[Herlédan Le Merdy, W. — to appear] *Unconditional foundations for supersingular isogeny-based cryptography*

Average hardness

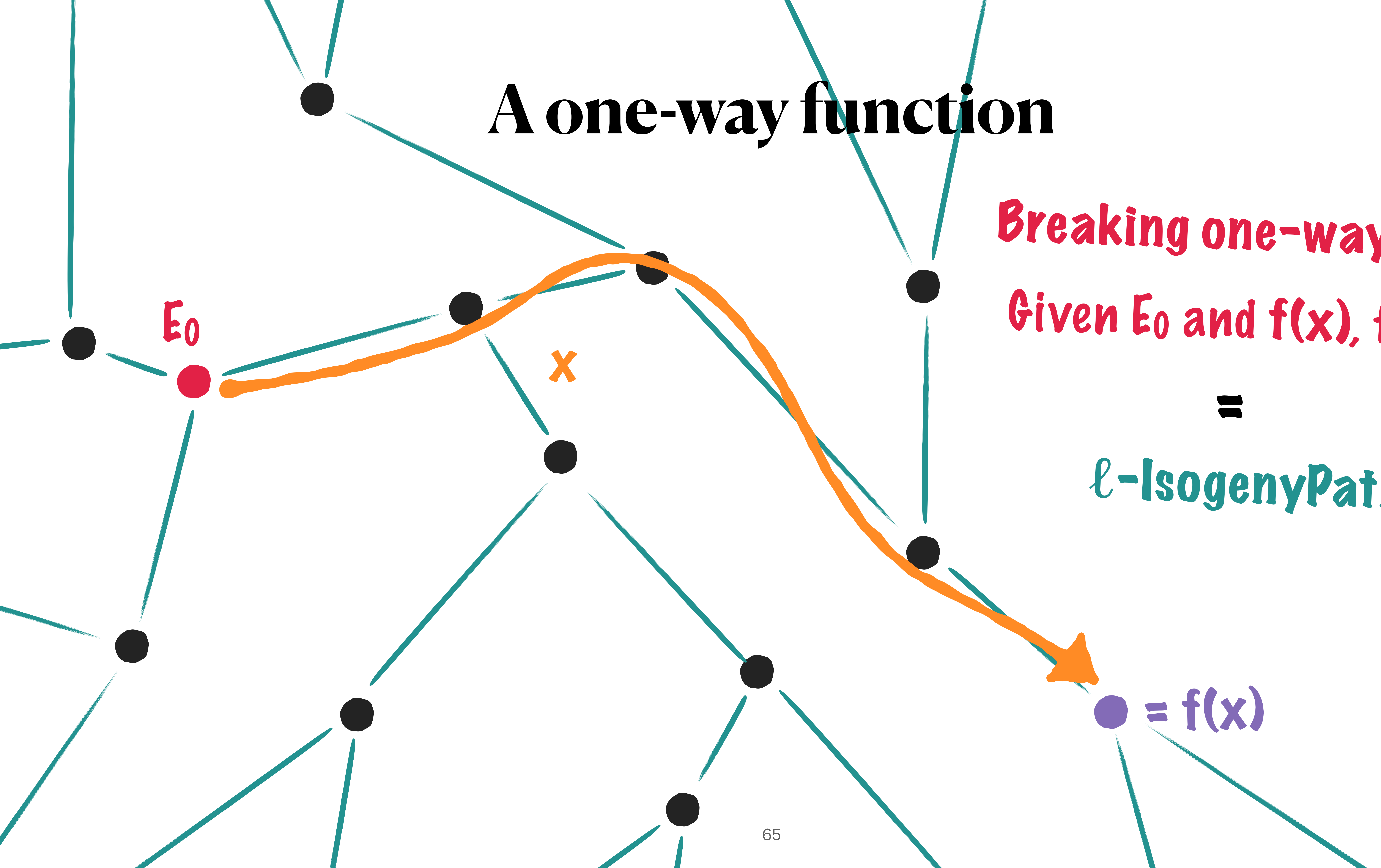
and worst-case to
average-case reductions

A one-way function

The CGL hash function

[Charles, Goren, Lauter – Journal of Cryptology 2009]

A one-way function



Breaking one-wayness:

Given E_0 and f(x), find x

=

ℓ-IsogenyPath

A one-way function

- **One-way function:** a function $f : X \to Y$ which is
  - ➡ **Easy to evaluate:** given $x \in X$, it is easy to compute $f(x)$
  - ➡ ~~**Hard to invert:** given $y \in Y$, it is hard to find some $x \in X$ such that $f(x) = y$~~
  - ➡ **Hard to invert:** let $x \in X$ uniformly random, and $y = f(x)$. There is no *efficient* algorithm A such that $A(y)$ outputs a preimage of y with *good probability*

For security, we care about average hardness

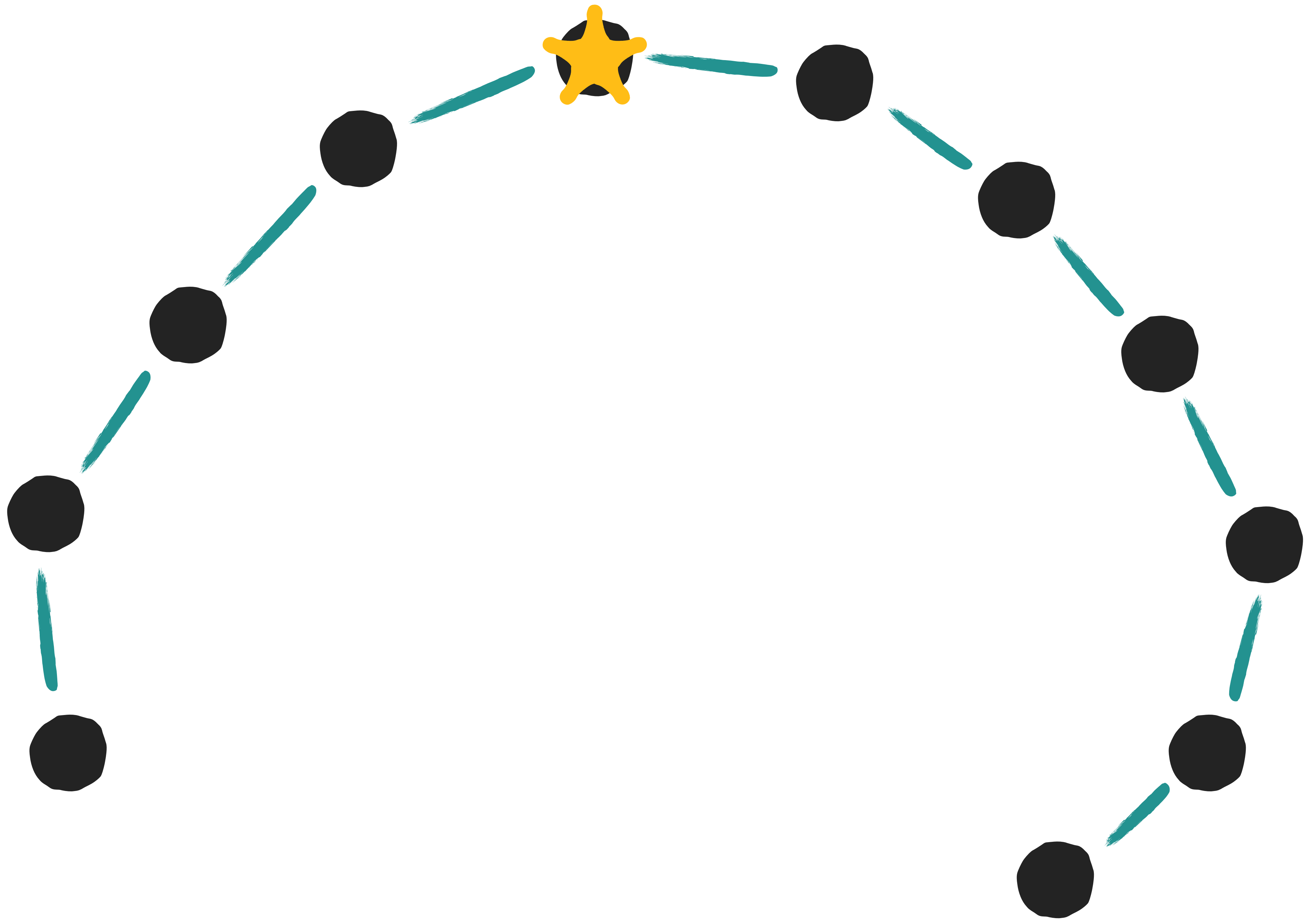
A problem should be hard on average for random inputs

A one-way function

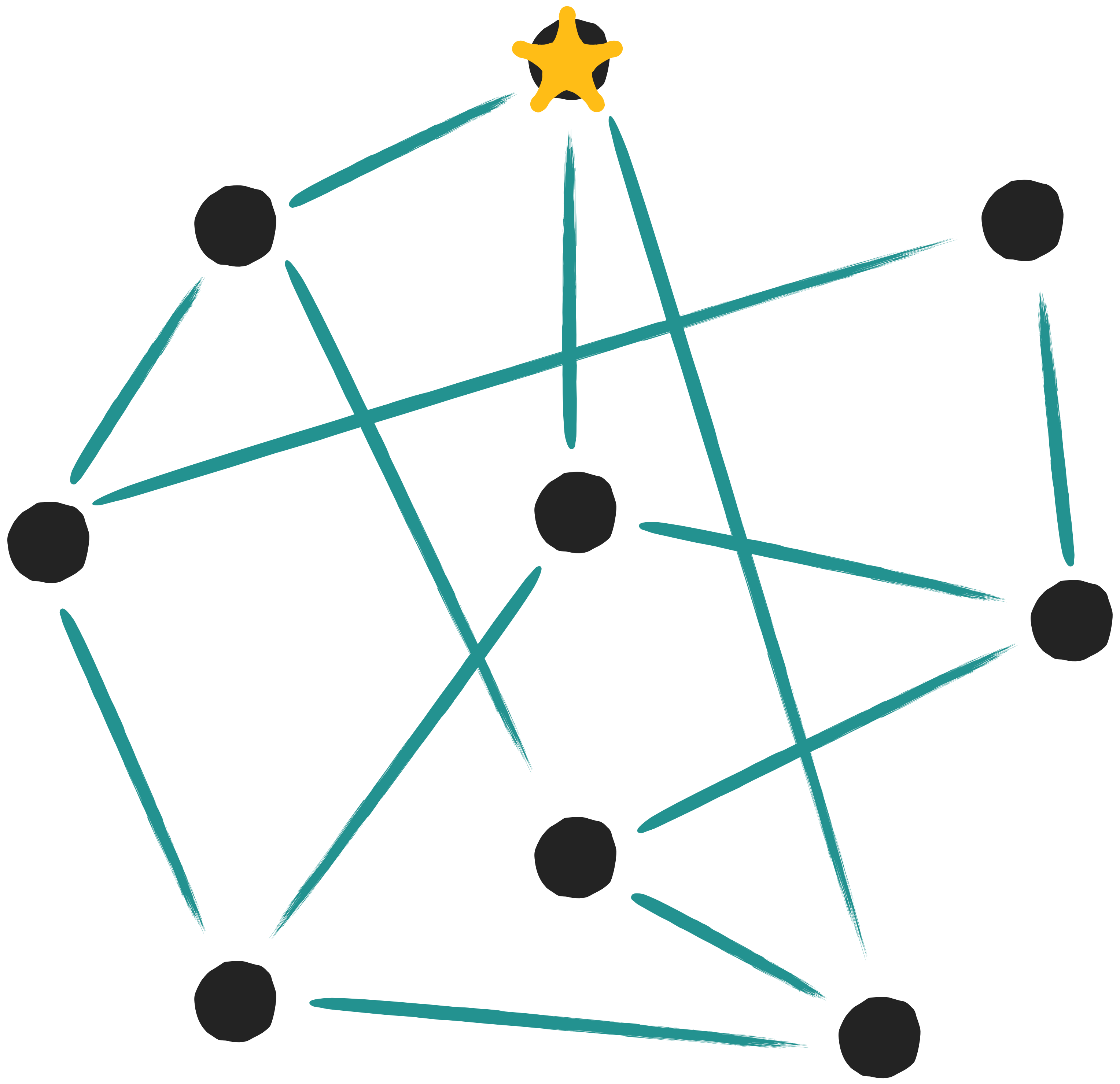
Rapid mixing

- Some graphs have better "mixing" properties than others...

Stays close to starting point for a long time...



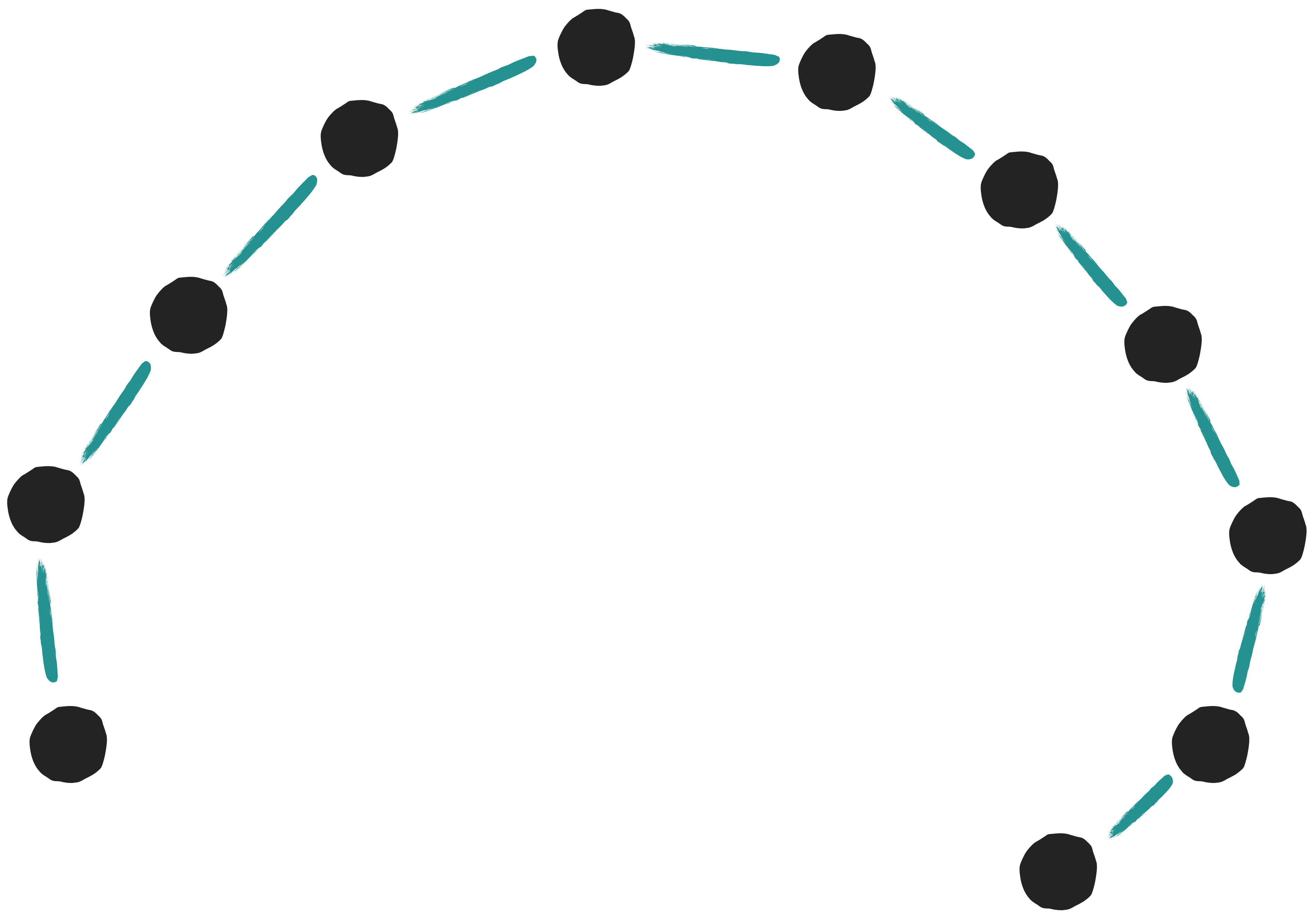
Rapidly goes anywhere



Rapid mixing

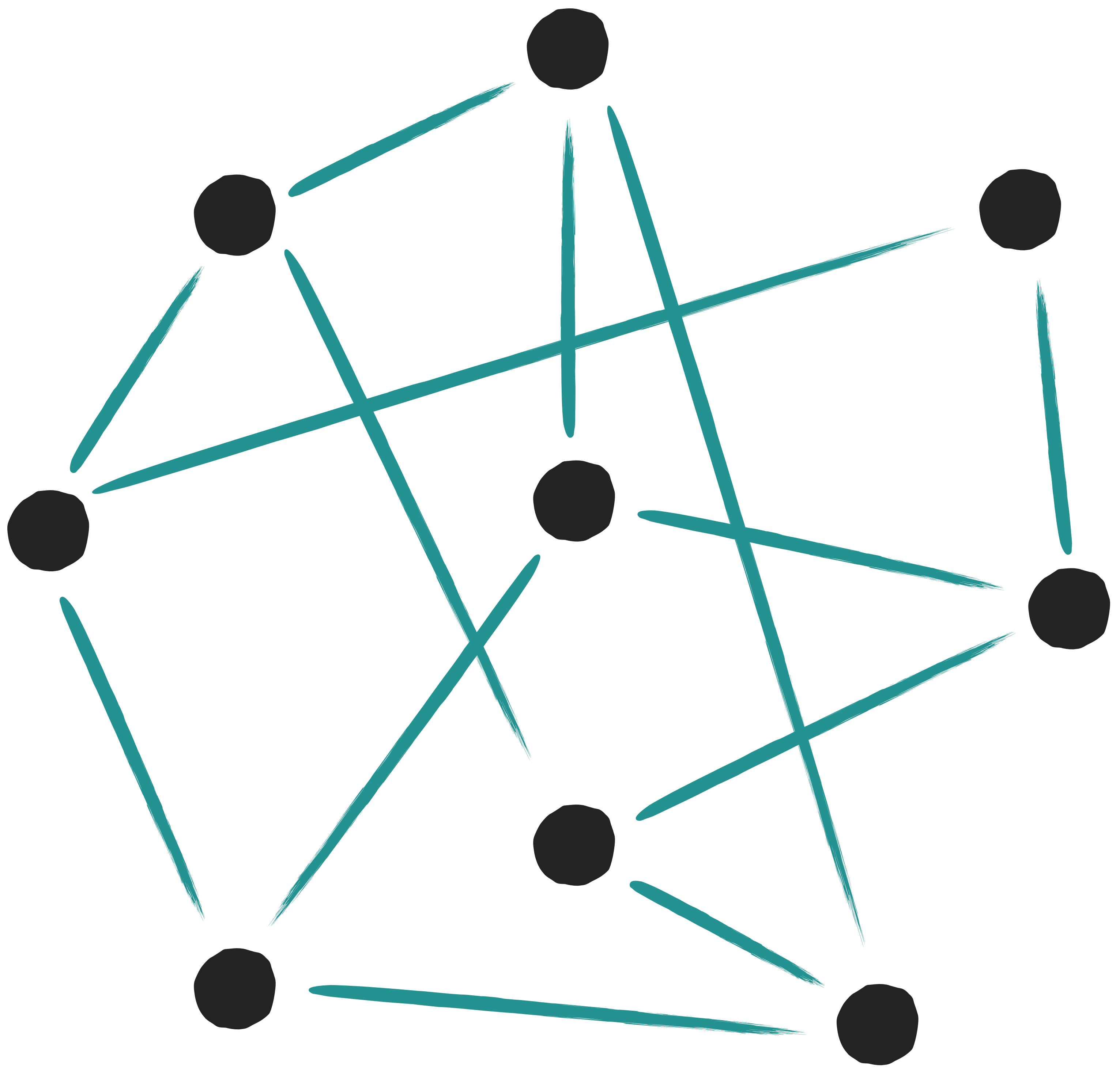
"slow mixing"

Stays close to starting point for a long time...



"rapid mixing"

Rapidly goes anywhere



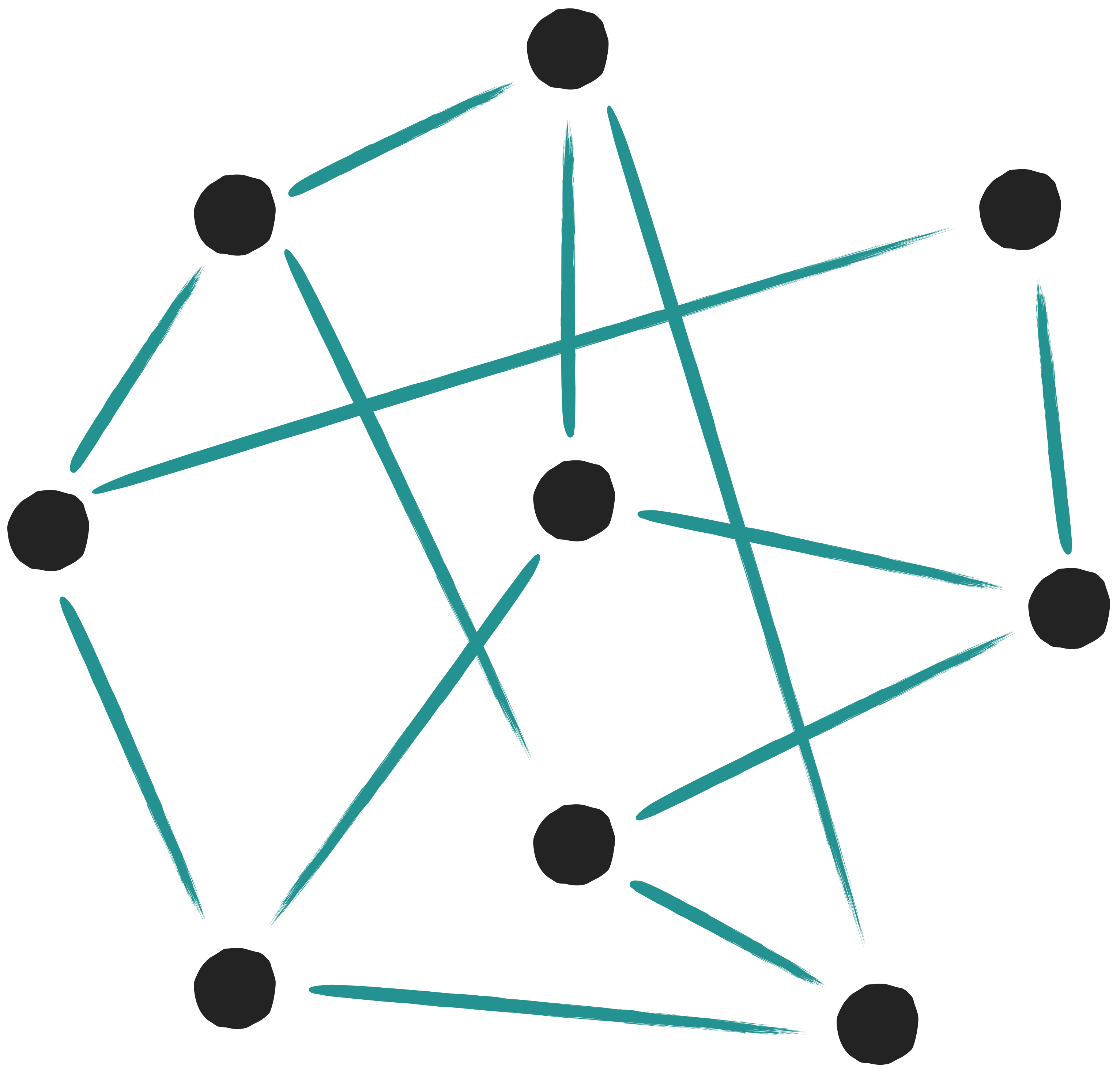
Rapid mixing

The best mixers are
Ramanujan graphs

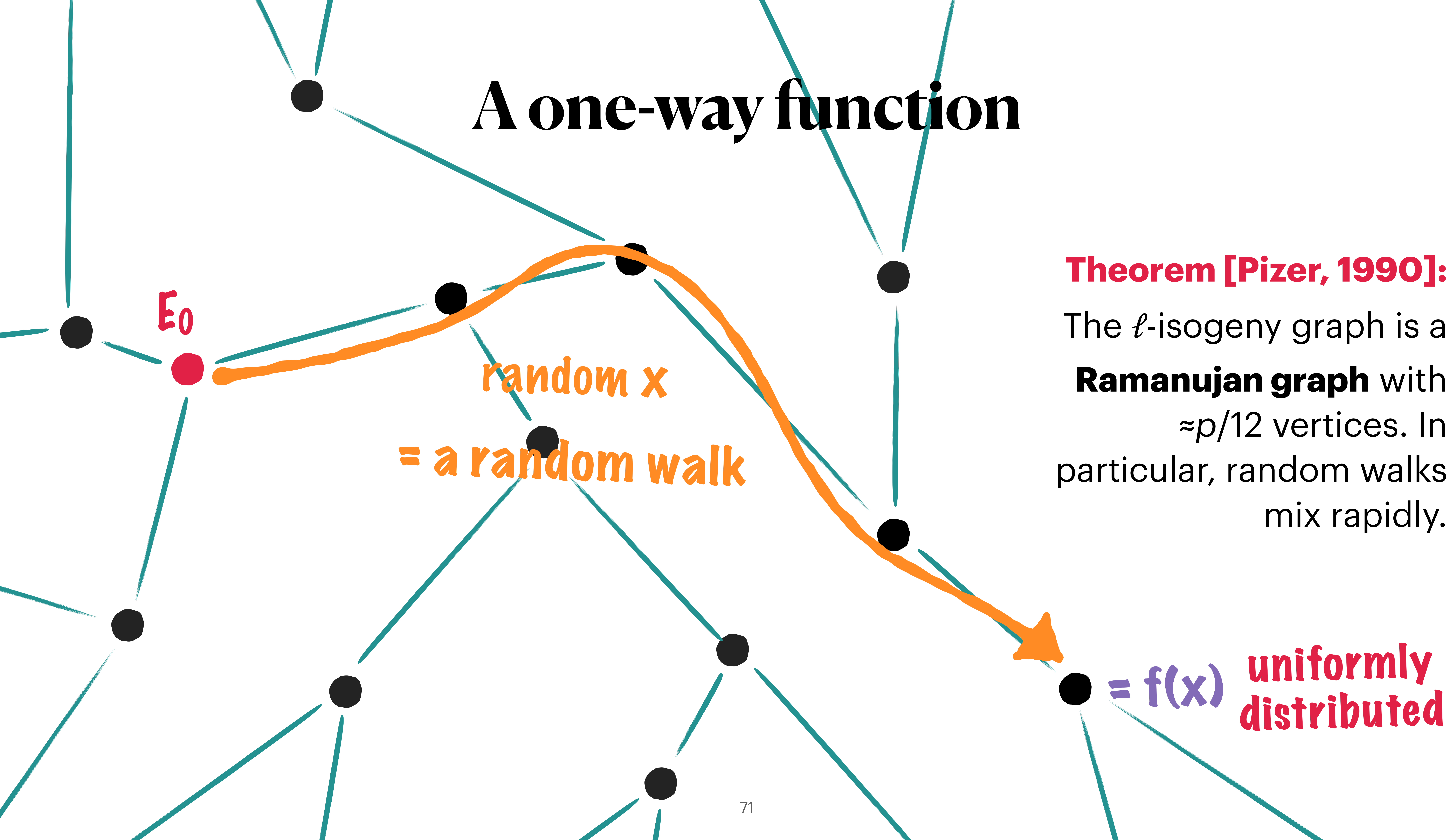
Theorem: In a **Ramanujan graph** with n vertices, a random walk of length ≈log(n) reaches a distribution **indistinguishable from uniform.**

"rapid mixing"

Rapidly goes anywhere



A one-way function

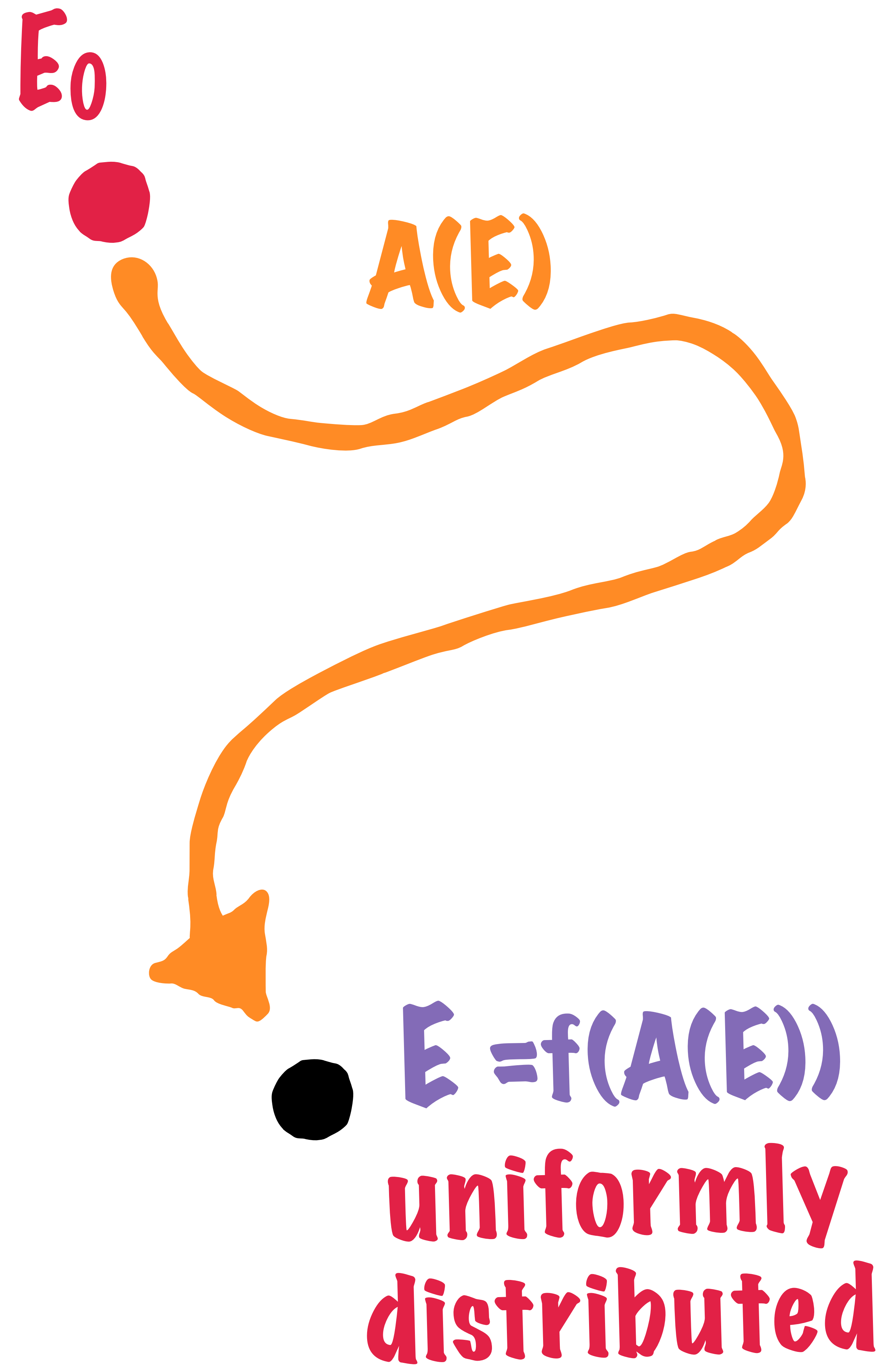


Theorem [Pizer, 1990]:

The ℓ-isogeny graph is a **Ramanujan graph** with $\approx p/12$ vertices. In particular, random walks mix rapidly.

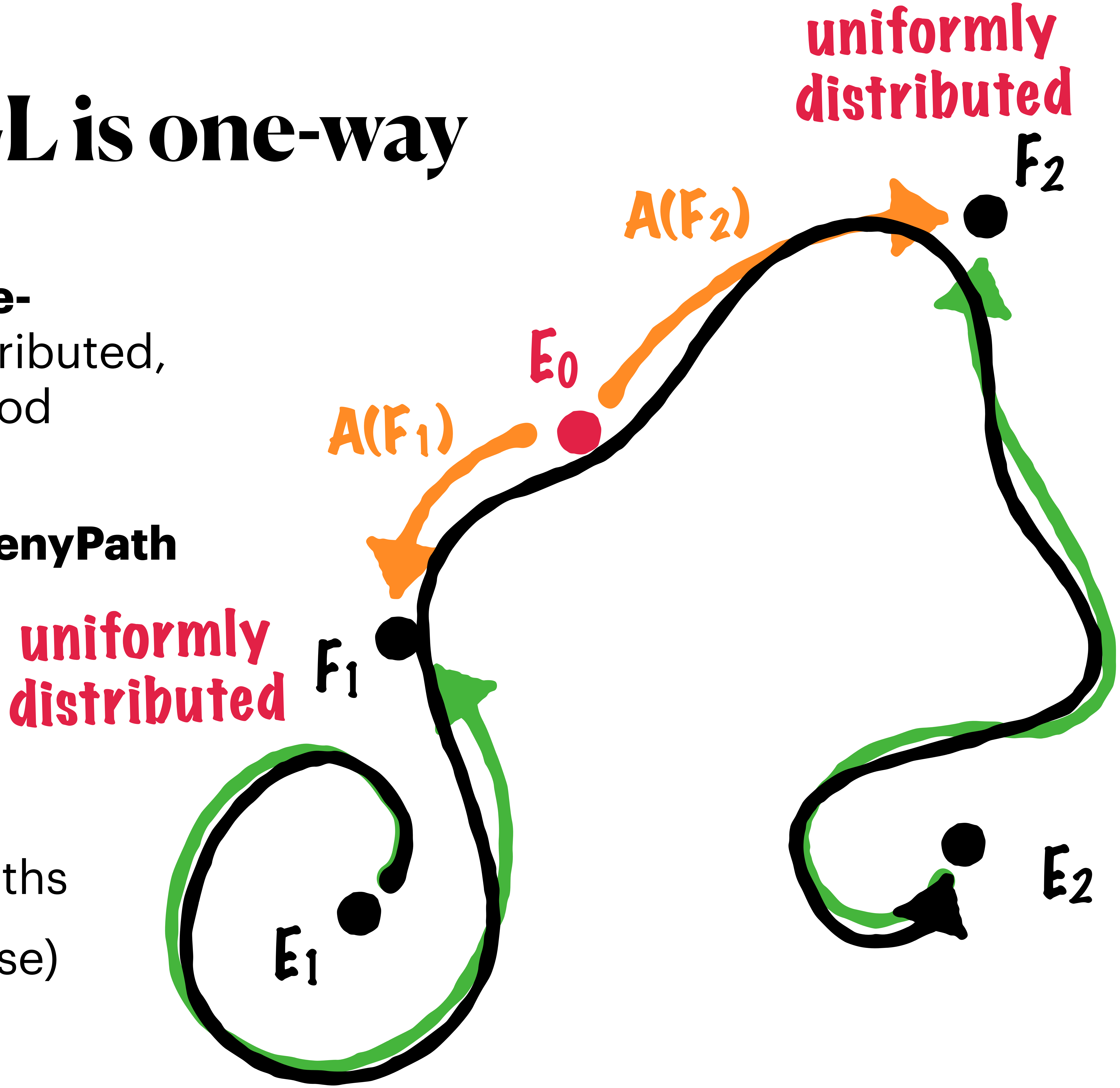
CGL is one-way

- Let **A** an algorithm **breaking one-wayness**: given E uniformly distributed, **A**(E) finds a path $E_0 \to E$ with good probability



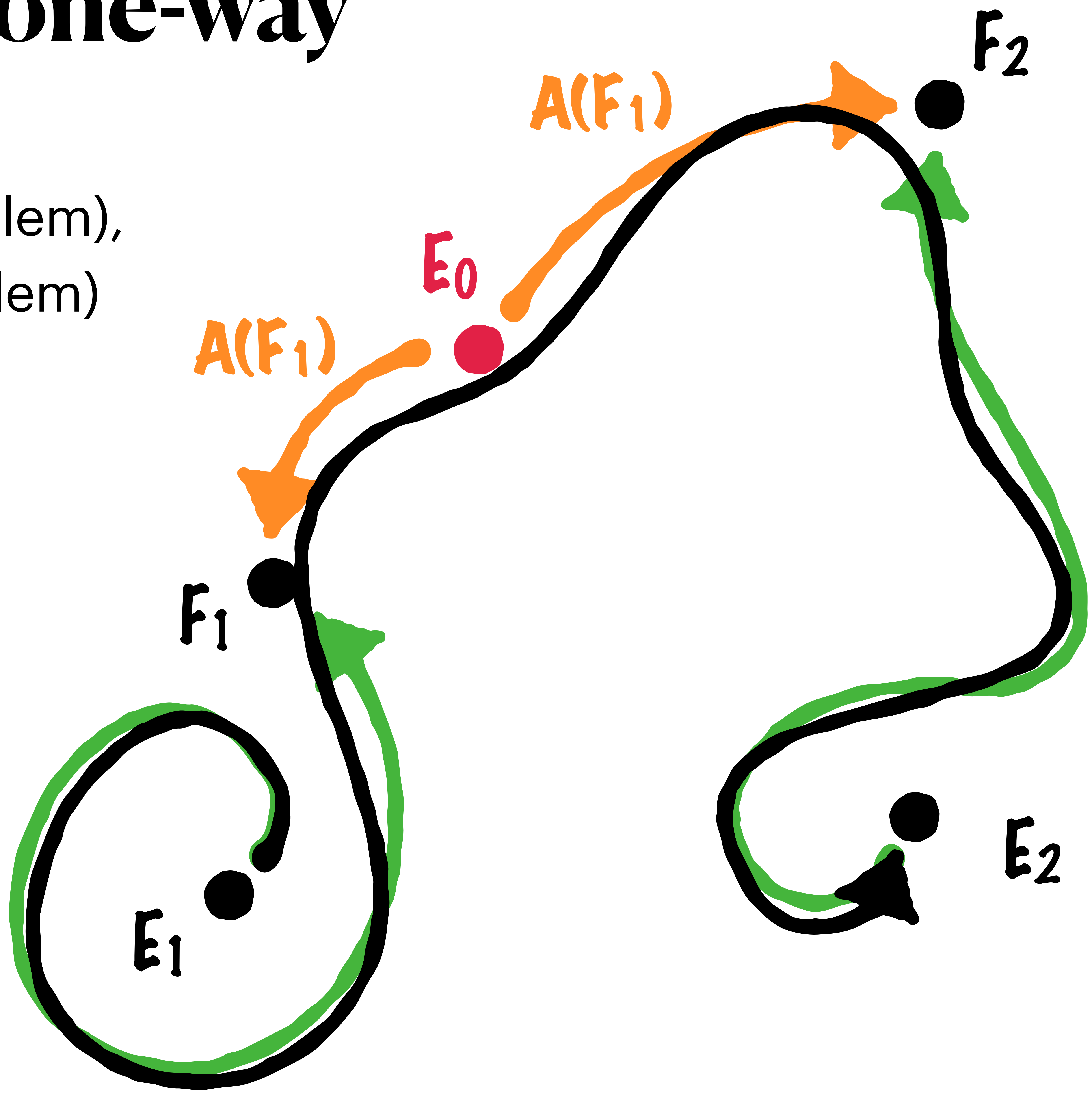
CGL is one-way

- Let **A** an algorithm **breaking one-wayness**: given E uniformly distributed, $\mathbf{A}(E)$ finds a path $E_0 \to E$ with good probability
- Let (E_1, E_2) an instance of **ℓ-IsogenyPath**
  1. Random walk $E_1 \to F_1$
  2. Call $\mathbf{A}(F_1)$
  3. Same for E_2...
  4. Return concatenation of paths
- Solves **ℓ-IsogenyPath** (worst case)



CGL is one-way

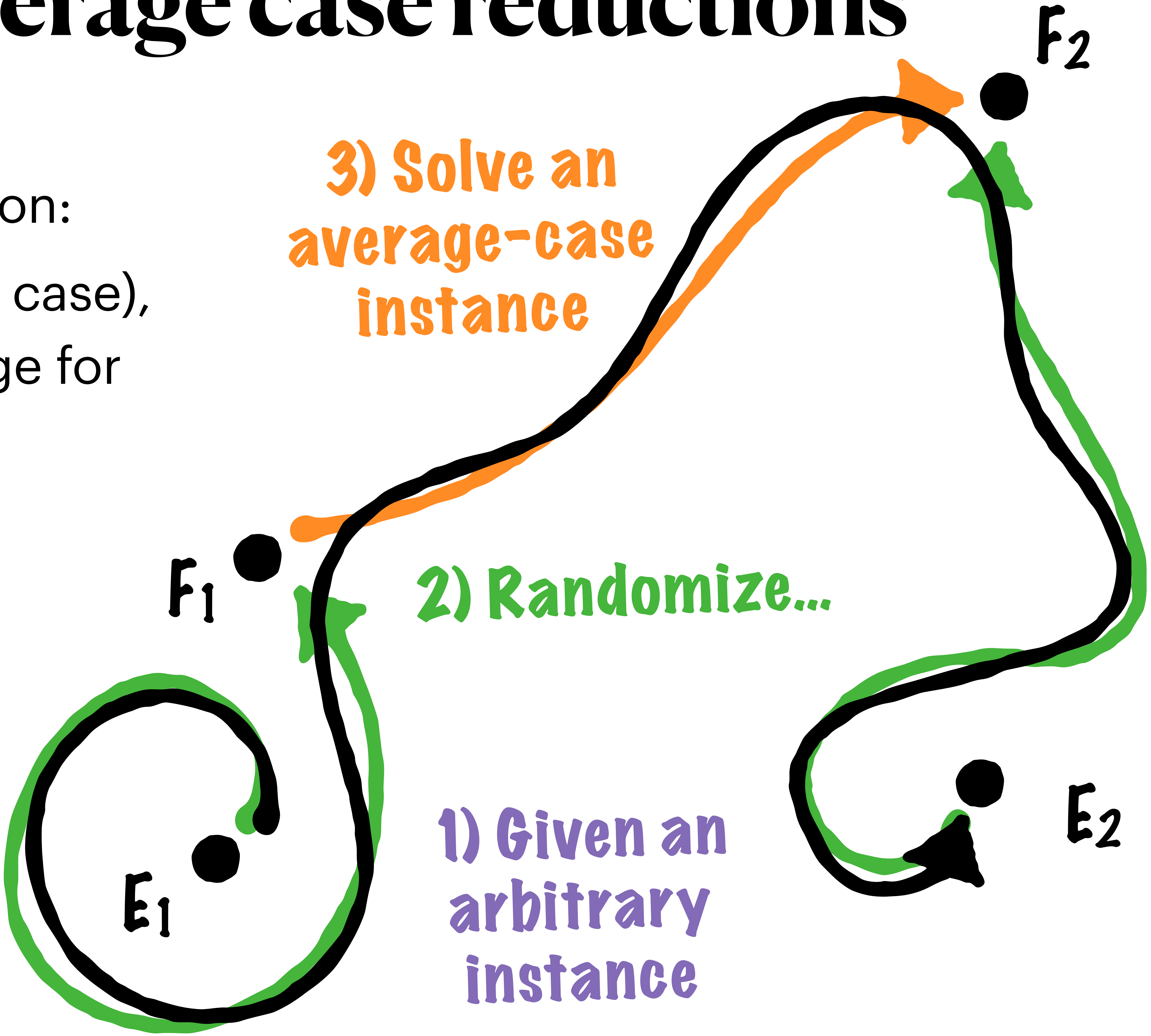
If ℓ-**IsogenyPath** is hard (worst case problem), then **CGL is one-way** (average-case problem)



Worst-case to average case reductions

A worst-case to average-case reduction:

If **ℓ-IsogenyPath** is hard (in the worst case), then **ℓ-IsogenyPath** is hard on average for uniformly random input



Which is hardest? Easiest?

EndRing

MaxOrder

ℓ-IsogenyPath

OneEnd

Isogeny

Assuming GRH, **if any of these is hard in the worst case, then all are hard on average!**

Without GRH, almost always true.

[Herlédan Le Merdy, W. — to appear] *Unconditional foundations for supersingular isogeny-based cryptography*

Solving ℓ-IsogenyPath

and Isogeny, EndRing, OneEnd, MaxOrder...

How hard are they?

EndRing

MaxOrder

ℓ-IsogenyPath

OneEnd

Isogeny

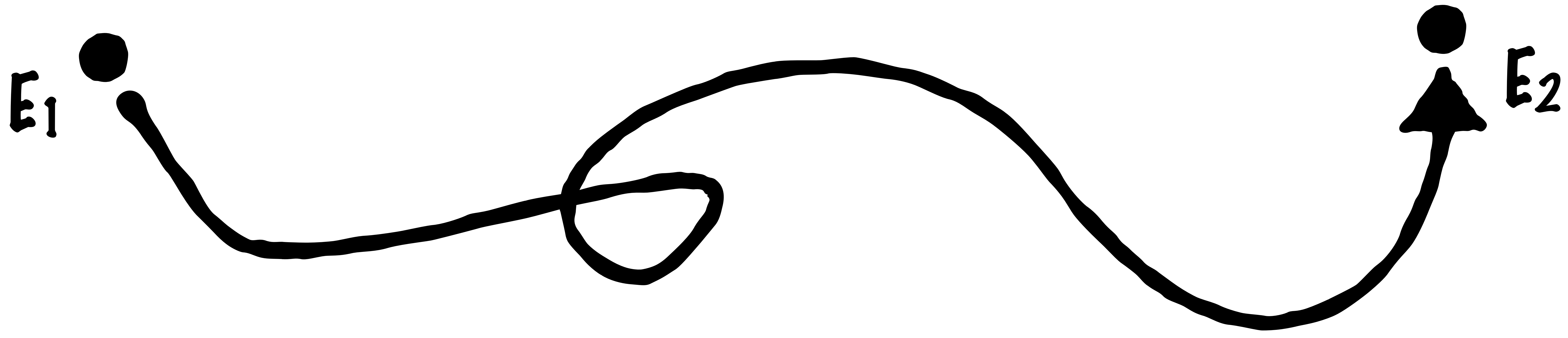
They are all as hard as each other...

But **how hard** is that?

Solving ℓ-IsogenyPath

The *ℓ-IsogenyPath* problem

Given E_1 and E_2 (supersingular) find an ℓ-isogeny path from E_1 to E_2

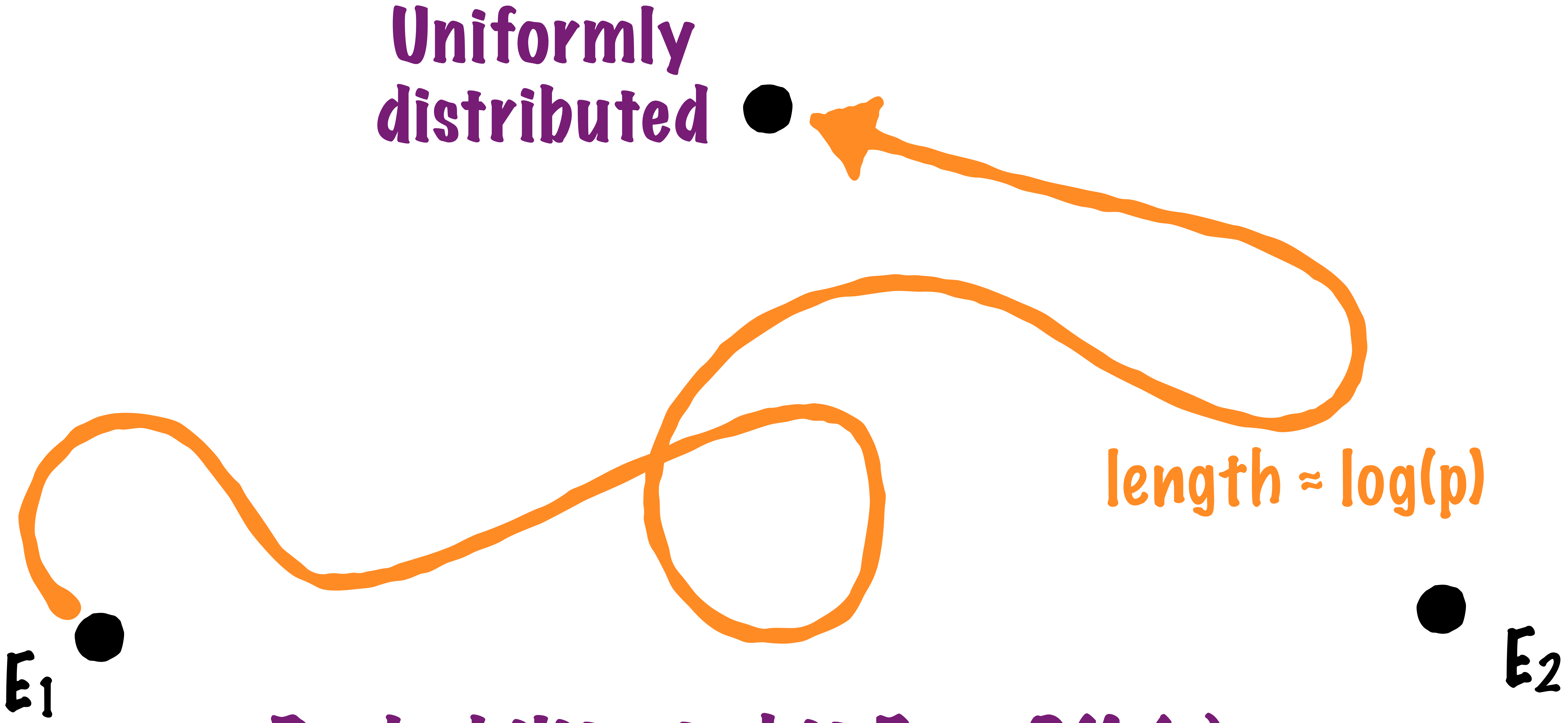


Solving ℓ-IsogenyPath

The supersingular ℓ-isogeny graph

- Approximately $p/12$ vertices
- Ramanujan

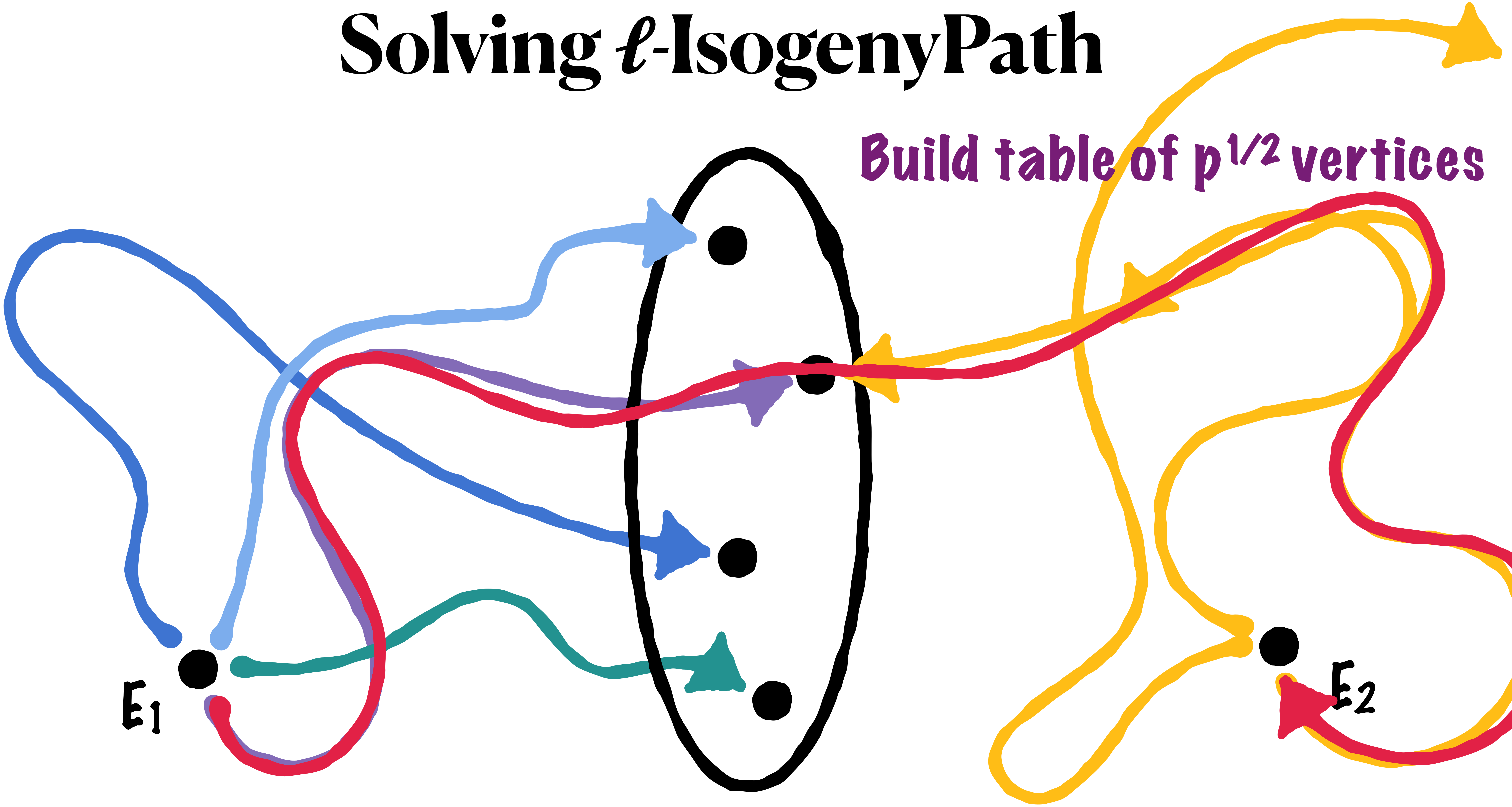
Solving ℓ-IsogenyPath



Probability to hit $E_2 = O(1/p)$

Success after $O(p)$ attempts...

Solving ℓ-IsogenyPath



Success after $O(p^{1/2})$ attempts!

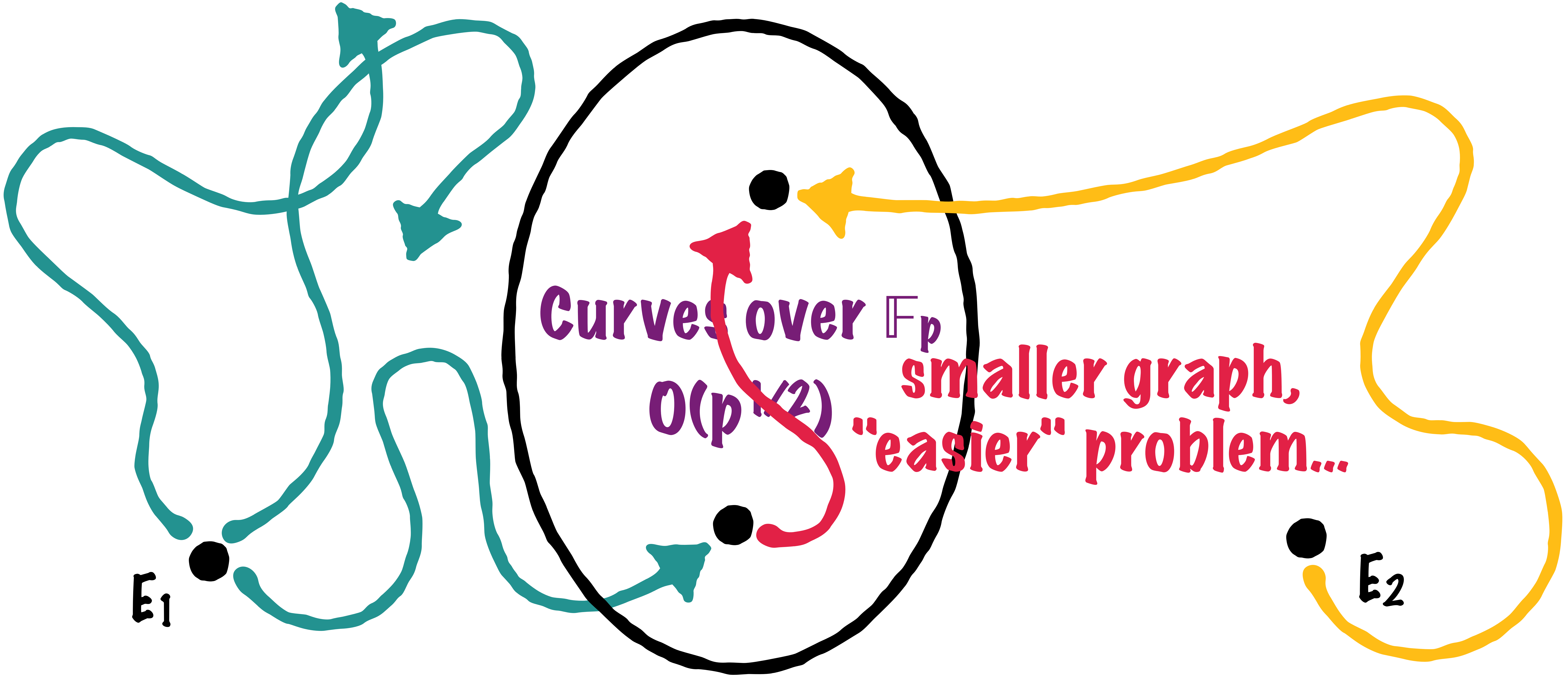
Solving ℓ-IsogenyPath

Theorem: There is an algorithm for **ℓ-IsogenyPath** in time $\tilde{O}(p^{1/2})$

Corollary: One can solve **Isogeny**, **EndRing**, **MaxOrder** and **OneEnd** in time $\tilde{O}(p^{1/2})$

Theorem [Delfs, Galbraith — DCC 2016]**:** There is an algorithm for **Isogeny** in time $\tilde{O}(p^{1/2})$ and space $\log(p)^{O(1)}$

Solving Isogeny



Success after $O(p^{1/2})$ attempts!

OneEnd = EndRing

OneEnd to find them all

Reducing EndRing to OneEnd

Suppose we have an oracle $\mathcal{O}$ solving **OneEnd**

Let E be an instance of **EndRing**: we wish to find generators of End(E)

Idea 0: *Sample until you make it…*

What if $\mathcal{O}(E)$ always returns the same α?

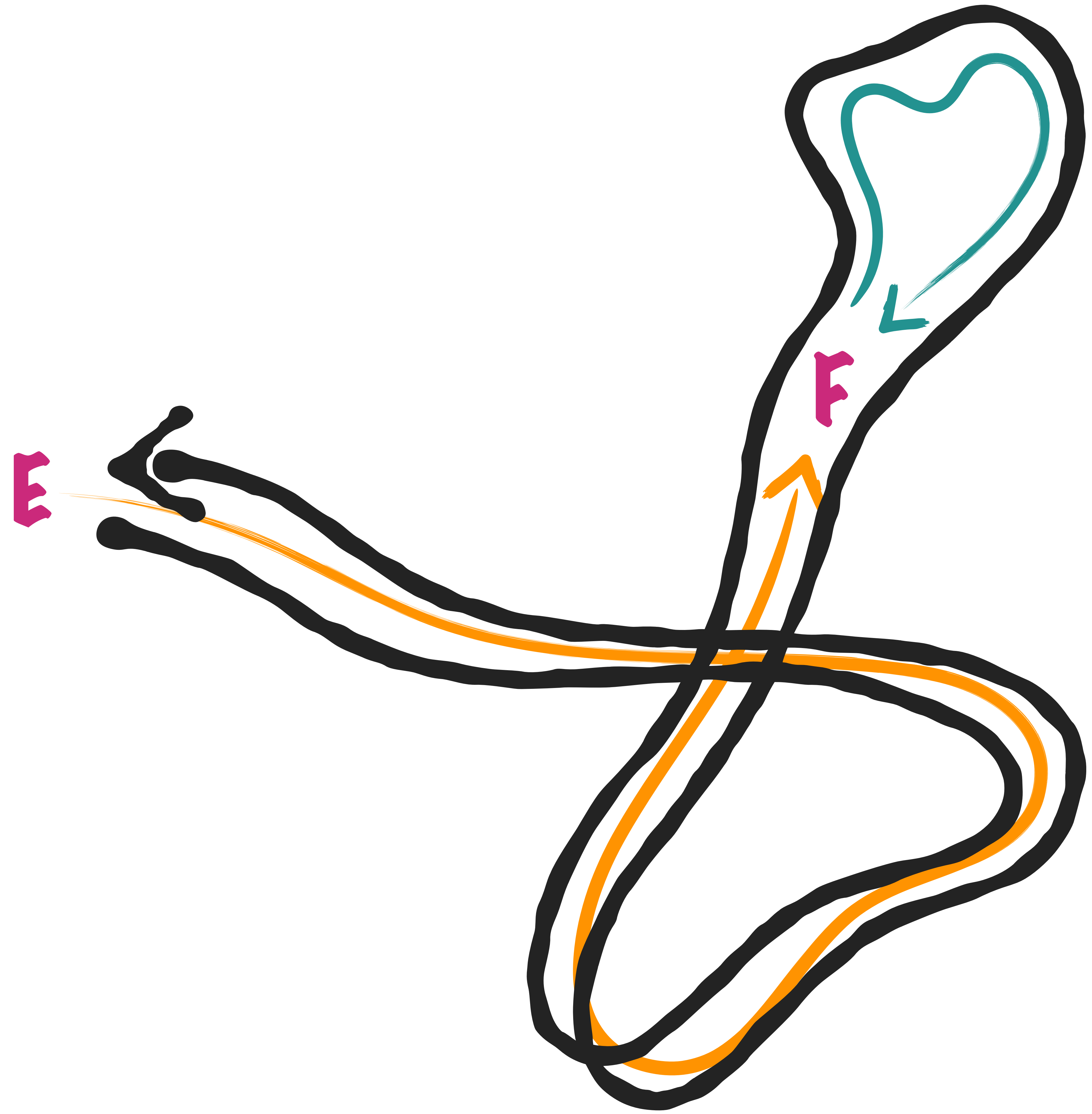
1. For i = 1, 2,… call $\mathcal{O}(E)$, which returns some $\alpha_i \in \mathrm{End}(E) \setminus \mathbb{Z}$
2. As soon as $(\alpha_i)_i$ generates End(E), extract a basis and return it

Efficient linear algebra!

Idea 1 [Eisenträger, Hallgren, Lauter, Morrison, Petit – Eurocrypt 2018]**:**

Randomize the oracle…

Enriching the oracle



Idea 1: Randomize the oracle

We construct a new oracle $\mathtt{Rich}^{\mathcal{O}}$

On input E:

1. Sample a random isogeny $\varphi : E \to F$
2. Call $\mathcal{O}(F)$ which returns $\alpha \in \mathrm{End}(F) \setminus \mathbb{Z}$
3. Return $\hat{\varphi} \circ \alpha \circ \varphi \in \mathrm{End}(E) \setminus \mathbb{Z}$

Reducing EndRing to OneEnd

Idea 1: Randomize the oracle

1. For $i = 1, 2, \ldots$ call $\mathtt{\textbf{Rich}}^{\mathcal{O}}(E)$, which returns some $\alpha_i \in \mathrm{End}(E) \setminus \mathbb{Z}$
2. As soon as $(\alpha_i)_i$ generates $\mathrm{End}(E)$, extract a basis and return it

Heuristic claim [Eisenträger, Hallgren, Lauter, Morrison, Petit – Eurocrypt 2018]**:**
$\mathrm{Rich}^{\mathcal{O}}$ is "random enough": it rapidly produces a generating set

Problem: It **fails**. There exist oracles $\mathcal{O}$ for which the algorithm does not terminate

Stabilization

Idea 2: Prove that the ring generated by $(\alpha_i)_i$ eventually stabilizes

Theorem 1: The probability distribution of $\mathtt{Rich}^{\mathcal{O}}(E)$ is stable under conjugation

In essence: any output α is as likely as any conjugate $\beta^{-1}\alpha\beta$

Theorem 2: Subrings of $\mathrm{End}(E)$ stable under conjugation are $\mathbb{Z} + M \cdot \mathrm{End}(E)$ for $M \in \mathbb{Z}$

Conclusion: The algorithm **eventually** generates a ring of the form $\mathbb{Z} + M \cdot \mathrm{End}(E)$

From a generating set of $\mathbb{Z}$ + $M \cdot \mathrm{End}(E)$, one can find a basis of $\mathrm{End}(E)$ 👍

"Eventually" = exponential time 👎

Stabilization

Idea 2: Prove that the ring generated by $(\alpha_i)_i$ eventually stabilizes

The tough part!

Theorem 1: The probability distribution of $\mathtt{Rich}^{\mathcal{O}}(E)$ is stable under conjugation

In essence: any output α is as likely as any conjugate $\beta^{-1}\alpha\beta$

Deuring correspondence

+

Theorem 2: Subrings of $\mathrm{End}(E)$ stable under conjugation are $\mathbb{Z} + M \cdot \mathrm{End}(E)$ for $M \in \mathbb{Z}$

Jacquet-Langlands correspondence

+

Conclusion: The algorithm **eventually** generates a ring of the form $\mathbb{Z} + M \cdot \mathrm{End}(E)$

Deligne's bound on coefficients of modular forms

From a generating set of $\mathbb{Z}$ + M·End(E), one can find a basis of End(E)

"Eventually" = exponential time

Stabilization

Theorem 1: The probability distribution of $\texttt{Rich}^{\mathcal{O}}(E)$ is stable under conjugation

Select E, call $\alpha \leftarrow \mathcal{O}(E)$, return (E, α) Random variable with distribution D_0

A "random walk operator" T on the space of probability distributions of (E, α)

Select E, a random isogeny $\varphi : E \to F$, call $\alpha \leftarrow \mathcal{O}(F)$, return $(E, \hat{\varphi} \circ \alpha \circ \varphi)$ $D_1 = T(D_0)$

- Long walk (i.e., large degree φ) $\Rightarrow$ $T(D_0)$ converges to a stationary distribution
- Stationary distribution $\Rightarrow$ stable under conjugation
- **Spectral analysis** of the operator T gives **convergence speed**

Stabilization

Theorem 1: The probability distribution of $\mathtt{Rich}^{\mathcal{O}}(E)$ is stable under conjugation

Deuring correspondence

Elliptic curves → Quaternions

The random walk operator is a Hecke operator on quaternionic automorphic forms

+

Jacquet-Langlands correspondence

Quaternions → Modular forms

Eigenvalues of the Hecke operator can be read off the coefficients of a classical modular form

+

Deligne's bound on coefficients of modular forms

Modular forms → … Modular forms

Bounds on coefficients imply bounds on eigenvalues of the random walk operator

Reducing EndRing to OneEnd

Outline of the reduction:

1. Initialize $S = \{ 1 \}$

2. While S does not generate a ring of the form $\mathbb{Z} + M \cdot \mathrm{End}(E)$, do:

  3. Sample $\alpha \leftarrow \mathtt{Rich}^{\mathcal{O}}(E)$

  4. $\alpha \leftarrow \mathtt{LazyReduce}(\alpha)$ (Idea 3)

  5. Add α to S

6. Extract from S a basis of $\mathrm{End}(E)$, and return it

Terminates! 👍

In exponential time... 👎

Faster stabilization

Idea 3: Stabilization can be made much faster by "reducing" each oracle output α_i.

Next problem: "Reducing" requires factoring large integers...

Idea 4: "Lazy reduction": do a partial factorization, and if something fails, it reveals a new factor

Reducing EndRing to OneEnd

Outline of the reduction:

1. Initialize $S = \{ 1 \}$
2. While S does not generate a ring of the form $\mathbb{Z} + M \cdot \mathrm{End}(E)$, do:
   3. Sample $\alpha \leftarrow \texttt{Rich}^{\mathcal{O}}(E)$
   4. $\alpha \leftarrow \texttt{LazyReduce}(\alpha)$ **(Idea 3)**
   5. Add α to S
6. Extract from S a basis of $\mathrm{End}(E)$, and return it

Polynomial time!

$$
\begin{tikzcd}
\text{EndRing} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \text{MaxOrder} \arrow[d, leftrightarrow] & \ell\text{-IsogenyPath} \arrow[l, "\text{GRH}"'] \\
\text{OneEnd} \arrow[r, leftrightarrow] & \text{Isogeny} \arrow[ur] &
\end{tikzcd}
$$