Introduction on Isogenies between Elliptic Curves

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Text books

The mathematical details of this presentation can be found in

[Sil09] J. H. Silverman, *The Arithmetic of Elliptic Curves*

[Was08] L. C. Washington, *Elliptic Curves: Number Theory and Cryptography*

Notation

- p is a **prime number** not equal to 2 or 3.
- q is a **power** of p.
- We only consider elliptic curves defined by

$$y^2 = x^3 + ax^2 + bx + c, \quad a, b, c \in \overline{\mathbb{F}}_p.$$

  If not specified, an elliptic curve is defined over $\mathbb{F}_q$.
- Elliptic curves are denoted by $E, E', E_1, E_2, \dots$
- The **neutral element** of an elliptic curve E is denoted by 0_E.
- For $P \in E$, the x-coordinate of P is denoted by $x(P)$ (similarly for $y(P)$).
- The **multiplication-by-n map** is denoted by $[n]$.

Isogeny (Definition)

Definition 1

Let E_1 and E_2 be elliptic curves.
An *isogeny* is a non-constant rational map

$$\varphi : E_1 \to E_2$$

such that $\varphi(0_{E_1}) = 0_{E_2}$.

Theorem 2 (Theorem III.4.8 in [Sil09])

Let $\varphi : E_1 \to E_2$ be an isogeny. Then φ is a **group homomorphism**, *i.e.,*

$$\varphi(P + Q) = \varphi(P) + \varphi(Q)$$

for all $P, Q \in E_1$.

Isogeny (Explicit form)

Since we consider elliptic curves defined by $y^2 = x^3 + ax^2 + bx + c$, we can write an **isogeny** φ in the form

$$\varphi(x, y) = \left(\frac{g_1(x)}{h_1(x)},\ y\frac{g_2(x)}{h_2(x)} \right),$$

where

- g_1, h_1, g_2, h_2 are polynomials over $\overline{\mathbb{F}}_p$,
- g_1 (resp. g_2) and h_1 (resp. h_2) have no common factors,
- h_1 and h_2 have the same roots.

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- g_1 (resp. g_2) and h_1 (resp. h_2) have no common factors,
- h_1 and h_2 have the same roots.

For $P \in E_1$,

$$\varphi(P) = 0_{E_2} \iff P = 0_{E_1} \text{ or } h_1(x(P)) = 0.$$

If g_1, h_1, g_2, h_2 are polynomials over $\mathbb{F}_{q^k}$, then we say φ is *defined over* $\mathbb{F}_{q^k}$.

Example (Scalar multiplication)

Let m be a nonzero integer. Then the **multiplication-by-m map**

$$[m] : E \to E$$

is an isogeny.

Example

Consider two elliptic curves E_1 and E_2:

$$\begin{aligned} E_1 : & y^2 = x^3 + ax^2 + bx, \\ E_2 : & y^2 = x^3 - 2ax^2 + (a^2 - 4b)x, \end{aligned}$$

where $a, b \in \mathbb{F}_q$ and $b(a^2 - 4b) \neq 0$.

The map $\varphi : E_1 \to E_2$ defined by

$$\varphi(x, y) = \left(\frac{x^2 + ax + b}{x},\ y\frac{b - x^2}{x^2} \right)$$

is an isogeny defined over $\mathbb{F}_q$.

Example (Frobenius map)

Let E be an elliptic curve defined by $y^2 = x^3 + ax^2 + bx + c$.

For an integer $k \geq 0$, we define an elliptic curve $E^{(p^k)}$ by

$$E^{(p^k)} : y^2 = x^3 + a^{p^k} x^2 + b^{p^k} x + c^{p^k}.$$

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Then the *p^k-th power Frobenius map* $\pi_{p^k} : E \to E^{(p^k)}$ defined by

$$\pi_{p^k}(x, y) = (x^{p^k}, y^{p^k})$$

is an isogeny.

Example (Frobenius map)

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is an isogeny.

Note:

$$y^{p^k} = y(x^3 + ax^2 + bx + c)^{(p^k - 1)/2}.$$

Isogeny theorem

Theorem 3 (Exercise 5.4 in [Sil09])

Let E_1 and E_2 be elliptic curves over $\mathbb{F}_q$.
Then the following are equivalent:

- *There exists an isogeny $\varphi : E_1 \to E_2$ defined over $\mathbb{F}_{q^k}$.*
- $\#E_1(\mathbb{F}_{q^k}) = \#E_2(\mathbb{F}_{q^k})$.

Isogeny theorem

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Let E_1 and E_2 be elliptic curves over $\mathbb{F}_q$.
Then the following are equivalent:

- *There exists an isogeny $\varphi : E_1 \to E_2$ defined over $\mathbb{F}_{q^k}$.*
- $\#E_1(\mathbb{F}_{q^k}) = \#E_2(\mathbb{F}_{q^k})$.

Remark

The latter statement does NOT mean $E_1(\mathbb{F}_{q^k}) \cong E_2(\mathbb{F}_{q^k})$ **as groups**.

E.g., There is an isogeny defined over $\mathbb{F}_7$ between

$$E_1 : y^2 = x^3 - x \quad \text{and} \quad E_2 : y^2 = x^3 + 4x.$$

Easy calculation shows that

$$E_1(\mathbb{F}_7) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \quad \text{and} \quad E_2(\mathbb{F}_7) \cong \mathbb{Z}/8\mathbb{Z}.$$

Degree of isogeny

Definition 4

Let $\varphi : E_1 \to E_2$ be an isogeny given by

$$\varphi(x,y) = \left(\frac{g_1(x)}{h_1(x)},\ y\frac{g_2(x)}{h_2(x)} \right).$$

The *degree* of φ is $\max\{\deg g_1, \deg h_1\}$ and is denoted by $\deg \varphi$.

Proposition 5

Let $\varphi : E_1 \to E_2$ and $\psi : E_2 \to E_3$ be isogenies. Then

$$\deg(\psi \circ \varphi) = \deg \psi \cdot \deg \varphi.$$

Degree of isogeny (Examples)

- $\deg \pi_{p^k} = p^k$.
- The isogeny defined by

$$\varphi(x, y) = \left(\frac{x^2 + ax + b}{x},\ y \frac{b - x^2}{x^2} \right)$$

is of degree 2.

Endomorphism

Definition 6

Let E be an elliptic curve. An *endomorphism* of E is

- an isogeny $\varphi : E \to E$
- or the zero map ($P \mapsto 0_E$ for all $P \in E$).

- $[n]$ is an endomorphism for all $n \in \mathbb{Z}$.
- $\pi_q : (x, y) \mapsto (x^q, y^q)$ is an endomorphism.
  ($\because$ E is defined over $\mathbb{F}_q \Rightarrow E = E^{(q)}$)

Endomorphism ring

Definition 7

The set of all **endomorphisms** of an elliptic curve E forms a **ring** under the point-wise addition and composition.

I.e., for endomorphisms α, β of E,

- $(\alpha + \beta)(P) := \alpha(P) + \beta(P)$ for all $P \in E$,
- $\alpha \cdot \beta := \alpha \circ \beta$.

We call this ring the *endomorphism ring* of E and denote it by $\mathrm{End}(E)$.

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Theorem 8 (Theorem III.9.3 and Theorem V.3.1 in [Sil09])

- E *is* **ordinary**

  $\Leftrightarrow \mathrm{End}(E) \cong$ *an order in an* **imaginary quadratic field***.*

- E *is* **supersingular**

  $\Leftrightarrow \mathrm{End}(E) \cong$ *a* **maximal** *order in a* **quaternion algebra***.*

Isomorphism

Definition 9

An *isomorphism* is an isogeny of degree 1.

Two elliptic curves E_1 and E_2 are *isomorphic*
if there is an isomorphism $\varphi : E_1 \to E_2$. We denote this by $E_1 \cong E_2$.

If φ is defined over $\mathbb{F}_{q^k}$, then we say E_1 and E_2 are *isomorphic over* $\mathbb{F}_{q^k}$.
We denote this by $E_1 \cong_{\mathbb{F}_{q^k}} E_2$.

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If φ is defined over $\mathbb{F}_{q^k}$, then we say E_1 and E_2 are *isomorphic over $\mathbb{F}_{q^k}$*.
We denote this by $E_1 \cong_{\mathbb{F}_{q^k}} E_2$.

Remark

If φ is an isomorphism, then φ is bijective.
However, the converse is NOT true in general.

E.g., the p-th power Frobenius map π_p is bijective but not an isomorphism.

Automorphism

Definition 10

An *automorphism* is an isomorphism from an elliptic curve to itself.

Definition 11

The set of all **automorphisms** of an elliptic curve E forms a **group** under the composition.

We call this group the *automorphism group* of E and denote it by $\mathrm{Aut}(E)$.

Note: $\mathrm{Aut}(E)$ is the unit group of $\mathrm{End}(E)$.

Automorphism group

Proposition 12 (Theorem III.10.1 and Corollary III.10.2 in [Sil09])

Let E be an elliptic curve.

1. $\mathrm{Aut}(E) = \{[\pm 1]\}$ *if* $j(E) \neq 0, 1728$.
2. $\mathrm{Aut}(E) \cong \mathbb{Z}/4\mathbb{Z}$ *if* $j(E) = 1728$.
3. $\mathrm{Aut}(E) \cong \mathbb{Z}/6\mathbb{Z}$ *if* $j(E) = 0$.

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3. $\mathrm{Aut}(E) \cong \mathbb{Z}/6\mathbb{Z}$ *if* $j(E) = 0$.

- For $E : y^2 = x^3 + x$ with $j(E) = 1728$,
$$(x, y) \mapsto (-x, \sqrt{-1}y)$$
generates $\mathrm{Aut}(E)$.

- For $E : y^2 = x^3 + 1$ with $j(E) = 0$,
$$(x, y) \mapsto (\zeta_3 x, -y)$$
generates $\mathrm{Aut}(E)$. (ζ_3 is a primitive 3rd root of unity in $\overline{\mathbb{F}}_p$.)

Separable isogeny (Definition)

Definition 13

Let $\varphi : E_1 \to E_2$ be an isogeny given by

$$\varphi(x,y) = \left(\frac{g_1(x)}{h_1(x)},\ y\frac{g_2(x)}{h_2(x)} \right).$$

We say φ is *separable* if $\frac{d}{dx}\frac{g_1(x)}{h_1(x)} \neq 0$ as a rational function, otherwise φ is *inseparable*.

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We say φ is *separable* if $\frac{d}{dx}\frac{g_1(x)}{h_1(x)} \neq 0$ as a rational function, otherwise φ is *inseparable*.

- The p^k-th power Frobenius map π_{p^k} is **inseparable**.
- The isogeny defined by

$$\varphi(x, y) = \left(\frac{x^2 + ax + b}{x},\ y\frac{b - x^2}{x^2} \right)$$

is **separable**.

Separable isogeny (Properties)

Proposition 14 (Corollary II.2.12 in [Sil09])

An isogeny $\varphi : E_1 \to E_2$ decomposes into a composition

$$E_1 \xrightarrow{\pi_{p^k}} E_1^{(p^k)} \xrightarrow{\psi} E_2,$$

where ψ is **separable**.

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Corollary 15

- *φ is* **inseparable** $\Leftrightarrow \dfrac{g_1(x)}{h_1(x)} = \dfrac{r(x^p)}{s(x^p)}$ *for some polynomials r, s.*
- *φ is* **inseparable** $\Rightarrow \deg \varphi \equiv 0 \pmod{p}$.

Kernel of isogeny (Definition)

Definition 16

Let $\varphi : E_1 \to E_2$ be an isogeny. The *kernel* of φ is

$$\ker \varphi = \{P \in E_1 \mid \varphi(P) = 0_{E_2}\}.$$

- $\ker[n] = E_1[n]$.
- $\ker \pi_{p^k} = \{0_{E_1}\}$.
- The kernel of the isogeny defined by

$$\varphi(x, y) = \left(\frac{x^2 + ax + b}{x},\ y\frac{b - x^2}{x^2}\right)$$

is $\{0_{E_1}, (0, 0)\}$.

Kernel of isogeny (Properties)

Proposition 17 (Theorem III.4.10 in [Sil09])

Let φ be an isogeny. Then

$$\#\ker\varphi \leq \deg\varphi.$$

If φ is **separable** *then $\#\ker\varphi = \deg\varphi$.*

Kernel of isogeny (Properties)

Proposition 17 (Theorem III.4.10 in [Sil09])

Let φ be an isogeny. Then

$$\# \ker \varphi \leq \deg \varphi.$$

If φ is **separable** *then $\# \ker \varphi = \deg \varphi$.*

Let φ be the isogeny defined by

$$\varphi(x, y) = \left(\frac{x^2 + ax + b}{x}, \ y \frac{b - x^2}{x^2} \right).$$

φ is separable, $\deg \varphi = 2$, and $\# \ker \varphi = \# \{0_{E_1}, (0, 0)\} = 2$.

Kernel of isogeny (Properties)

Proposition 18 (Proposition III.4.12 in [Sil09])

Let E be an elliptic curve and G be a finite subgroup of E.
Then there exist a unique (up to isomorphism) E' and a **separable** *isogeny*

$$\varphi : E \to E'$$

such that $\ker \varphi = G$. (E' and φ are not necessarily defined over $\mathbb{F}_q$.)

Kernel of isogeny (Properties)

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Let E be an elliptic curve and G be a finite subgroup of E.
Then there exist a unique (up to isomorphism) E' and a **separable** *isogeny*

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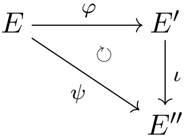
such that $\ker \varphi = G$. (E' and φ are not necessarily defined over $\mathbb{F}_q$.)

"up to isomorphism" means:
E'' and ψ satisfy the same conditions $\Rightarrow$ there is an **isomorphism** ι s.t.



We denote E' by E/G.

Proposition 19 (Remark III.4.13.2 in [Sil09])

In the previous proposition, suppose that G is invariant under **the q^k-th power Frobenius map** π_{q^k}*, i.e.,*

$$\pi_{q^k}(P) \in G \quad \text{for all } P \in G.$$

Then there exist a unique (up to isomorphism over $\mathbb{F}_{q^k}$) E' defined over $\mathbb{F}_{q^k}$ and a separable isogeny

$$\varphi : E \to E'$$

defined over $\mathbb{F}_{q^k}$ such that $\ker \varphi = G$.

Equivalence of isogenies

Definition 20

Two separable isogenies φ_1 and φ_2 are *equivalent* if $\ker \varphi_1 = \ker \varphi_2$.

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Two separable isogenies φ_1 and φ_2 are *equivalent* if $\ker \varphi_1 = \ker \varphi_2$.

Let φ_1 and φ_2 be equivalent isogenies with the same codomain.

$$E_1 \underset{\varphi_2}{\overset{\varphi_1}{\rightrightarrows}} E_2$$

By Proposition 18, $\exists \iota \in \mathrm{Aut}(E_2)$ such that $\varphi_1 = \iota \circ \varphi_2$.

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Two separable isogenies φ_1 and φ_2 are *equivalent* if $\ker\varphi_1 = \ker\varphi_2$.

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By Proposition 18, $\exists\iota \in \mathrm{Aut}(E_2)$ such that $\varphi_1 = \iota \circ \varphi_2$.

More explicitly, one of the following holds:

- $\varphi_1 = \varphi_2$ or $\varphi_1 = -\varphi_2$.
- $j(E_2) = 1728$ and $\varphi_1 = \iota \circ \varphi_2$ for $\iota \in \mathrm{Aut}(E_2)$ of order 4.
- $j(E_2) = 0$ and $\varphi_1 = \iota \circ \varphi_2$ for $\iota \in \mathrm{Aut}(E_2)$ of order 3 or 6.

Dual isogeny

Theorem 21 (Theorem III.6.1 in [Sil09])

Let $\varphi : E_1 \to E_2$ *be an isogeny of degree* m.
Then there is a unique isogeny

$$\hat{\varphi} : E_2 \to E_1 \ \ \textit{such that} \ \ \hat{\varphi} \circ \varphi = [m].$$

We call $\hat{\varphi}$ the *dual isogeny* of φ and always use the notation $\hat{\varphi}$ for it.

"Unique" means that $\hat{\varphi}$ is literally unique.

Dual isogeny

Proposition 22 (Theorem III.6.2 in [Sil09])

Let $\varphi : E_1 \to E_2$ *be an isogeny.*

1. *For another isogeny* $\psi : E_2 \to E_3$*,*
$$\widehat{\psi \circ \varphi} = \hat{\varphi} \circ \hat{\psi}.$$
2. *For another isogeny* $\lambda : E_1 \to E_2$*,*
$$\widehat{\varphi + \lambda} = \hat{\varphi} + \hat{\lambda}.$$
3. *For all* $m \in \mathbb{Z} \setminus \{0\}$*,*
$$\widehat{[m]} = [m] \text{ and } \deg[m] = m^2.$$
4. $\deg \hat{\varphi} = \deg \varphi.$
5. $\hat{\hat{\varphi}} = \varphi.$

Dual isogeny

Remark

Let φ_1 and φ_2 be **equivalent** isogenies with the same codomain.

$$E_1 \underset{\varphi_2}{\overset{\varphi_1}{\rightrightarrows}} E_2$$

If $j(E_2) = 0$ or 1728 and $E_1 \not\cong E_2$, then $\hat{\varphi}_1$ and $\hat{\varphi}_2$ could be **NOT equivalent**.

Dual isogeny

Remark

Let φ_1 and φ_2 be **equivalent** isogenies with the same codomain.

$$E_1 \underset{\varphi_2}{\overset{\varphi_1}{\rightrightarrows}} E_2$$

If $j(E_2) = 0$ or 1728 and $E_1 \not\cong E_2$, then $\hat{\varphi}_1$ and $\hat{\varphi}_2$ could be **NOT equivalent**.

Example:
Suppose $j(E_2) = 1728$ and let $\iota \in \mathrm{Aut}(E_2)$ of order 4.
An separable isogeny $\varphi : E_1 \to E_2$ and $\iota \circ \varphi$ are **equivalent**.

$$\ker \widehat{\iota \circ \varphi} = \ker(\hat{\varphi} \circ \hat{\iota}) = \hat{\iota}^{-1}(\ker \hat{\varphi}) \neq \ker \hat{\varphi} \text{ in general.}$$

So $\hat{\varphi}$ and $\widehat{\iota \circ \varphi}$ are **NOT equivalent** in general.

Decomposition of isogeny

Proposition 23

Let $\varphi : E_1 \to E_2$ *be a separable isogeny of degree* $m_1 m_2$*.*
Then φ *can be decomposed into*

$$E_2 \xrightarrow{\varphi_1} E_3 \xrightarrow{\varphi_2} E_2,$$

where $\deg \varphi_1 = m_1$ *and* $\deg \varphi_2 = m_2$*.*

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where $\deg \varphi_1 = m_1$ *and* $\deg \varphi_2 = m_2$*.*

(**Sketch of proof**)
$G := \ker \varphi$ contains a subgroup G_1 of order m_1.
$\exists \varphi_1 : E_1 \to E_3$ such that $\ker \varphi_1 = G_1$ (Proposition 18).
$\exists \varphi_2 : E_3 \to E_4$ such that $\ker \varphi_2 = \varphi_1(G)$ (Proposition 18).
Then $\ker(\varphi_2 \circ \varphi_1) = \varphi_1^{-1}(G) = G_1 + G = G$.
Thus, there is an isomorphism $\iota : E_4 \to E_2$ such that $\varphi = \iota \circ \varphi_2 \circ \varphi_1$. □

Isogeny of degree p

Proposition 24 (Corollary III.6.4 and Theorem V.3.1 in [Sil09])

- $E[m] \cong \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$ *for* $m \not\equiv 0 \pmod p$.
- $E[p] \cong \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{if } E \text{ is ordinary,} \\ \{0_E\} & \text{if } E \text{ is supersingular.} \end{cases}$

Isogeny of degree p

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- $E[p] \cong \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{if } E \text{ is ordinary,} \\ \{0_E\} & \text{if } E \text{ is supersingular.} \end{cases}$

Corollary 25

- *If* E *is* **ordinary**, *there are exactly two isogenies of degree* p *from* E,
  1. π_p
  2. *the separable isogeny of kernel* $E[p]$.
- *If* E *is* **supersingular**, *only* π_p *is the isogeny of degree* p *from* E.

Cyclic isogeny

Proposition 26

Let $\varphi : E_1 \to E_2$ *be a separable isogeny.*
Then there exists an integer m *such that* φ *can be decomposed into*

$$E_1 \xrightarrow{[m]} E_1 \xrightarrow{\varphi_1} E_2,$$

where $\ker \varphi_1$ *is cyclic.*

Cyclic isogeny

Proposition 26

Let $\varphi : E_1 \to E_2$ *be a separable isogeny.*
Then there exists an integer m *such that* φ *can be decomposed into*

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where $\ker \varphi_1$ *is cyclic.*

Sketch of proof

From the structure theorem of finite abelian groups,

$$\ker \varphi \cong \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z} \quad (m \mid n).$$

Therefore, φ can be decomposed into

$$E \xrightarrow{[m]} E \xrightarrow{\varphi_1} E_1,$$

where $\ker \varphi_1 = [m] \ker \varphi \cong \mathbb{Z}/(n/m)\mathbb{Z}$. □

Cyclic isogeny

Definition 27

Let m be a positive integer.
An *m-isogeny* is a **separable** isogeny with **cyclic** kernel of order m.

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Theorem 28

Let m be a positive integer coprime with p.
*Then the number of m-**isogenies** from E is*

$$m \prod_{\ell} \left(1 + \frac{1}{\ell}\right),$$

where the product is taken over all prime divisors ℓ of m.

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Let m be a positive integer.
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*Then the number of m-**isogenies** from E is*

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where the product is taken over all prime divisors ℓ of m.

Sketch of proof

Consider the number of cyclic subgroups of order m in $\mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$.

Algorithm

Computing isogenies

Our task

Given an **elliptic curve** E and a **finite subgroup** G of E,
compute the **codomain** E' of a **separable isogeny** φ with kernel G.

In addition, given a point P on E, compute $\varphi(P)$.

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Note:

- It is enough to consider separable isogenies.
  $\because$ An inseparable is decomposed into a separable isogeny
  and a Frobenius isogeny. (Frobenius isogenies are easy to compute.)

- We can assume that G is cyclic.
  $\because$ Otherwise, φ is decomposed into a scalar multiplication
  and an isogeny with a cyclic kernel.

Theorem 29 (Vélu's Formula, Theorem 12.16 in [Was08])

Let E be an elliptic curve defined by

$$y^2 = x^3 + a_2x^2 + a_4x + a_6 =: f(x),$$

and G be a finite subgroup of E.
The following E' and φ give an isogeny $\varphi : E \to E'$ with kernel G.

$$E' : y^2 = x^3 + a_2x^2 + (a_4 - 5v)x + a_6 - 4a_2v - 7w,$$
$$\varphi(x, y) = \left(F(x),\ y \cdot F'(x)\right),$$

where $v = \sum_{P \in G \setminus \{0_E\}} f'(x(P)),\ w = \sum_{P \in G \setminus \{0_E\}} \left(2f(x(P)) + xf'(x(P))\right),$

$$F(x) = x + \sum_{P \in G \setminus \{0_E\}} \left(\frac{f'(x(P))}{x - x(P)} + \frac{2f(x(P))}{(x - x(P))^2} \right).$$

For a rational function $r(x)$, $r'(x)$ denotes the derivative of $r(x)$.

Remarks on Vélu's Formula

- Vélu's formula requires $O(\#G)$ operations.
- We do NOT need the y-coordinate of the points in G.
  $\because G = -G$.
- The operations in the computation are on a field containing the x-coordinates of the points in G.

  I.e., the operations are on $\mathbb{F}_{q^k}$ such that

  $$\pi_{q^k}(P) = P \text{ or } -P \text{ for all } P \in G.$$

  Note: φ could be defined over a smaller field than $\mathbb{F}_{q^k}$.
- In practice, we often use Montgomery curves, which have more efficient formulas for isogenies (see Appendix).

Chain of isogenies

Let G be a **cyclic** subgroup of E of order n and φ be the separable isogeny with kernel G.

Chain of isogenies

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Assume that $n = \prod_{i=1}^{k} \ell_i$ for primes ℓ_i (not necessarily distinct). From Proposition 23, we can decompose φ into a chain of isogenies

$$E \xrightarrow{\varphi_1} E_1 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_k} E_k$$

where $\deg \varphi_i = \ell_i$.

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In many cases,
computing φ_i's sequentially is **more efficient** than computing φ directly.

$\because$ The cost of computing φ is linear in $n = \prod_{i=1}^{k} \ell_i$,
while the cost of computing all φ_i's is linear in $\sum_{i=1}^{k} \ell_i$.

Computing a chain of isogenies

We consider computing a chain of isogenies

$$E \xrightarrow{\varphi_1} E_1 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_k} E_k$$

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$$E \xrightarrow{\varphi_1} E_1 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_k} E_k$$

where $\deg \varphi_i = \ell_i$.

Since the kernel G of the composite isogeny is cyclic, we have

$$\begin{aligned}
\ker \varphi_1 &= [n/\ell_1]G, \\
\ker \varphi_2 &= [n/(\ell_1\ell_2)]\varphi_1(G), \quad (\because \#\varphi_1(G) = n/\ell_1), \\
&\vdots \\
\ker \varphi_i &= [n/(\ell_1 \cdots \ell_i)]\varphi_{i-1} \circ \cdots \circ \varphi_1(G), \\
&\vdots \\
\ker \varphi_k &= \varphi_{k-1} \circ \cdots \circ \varphi_1(G).
\end{aligned}$$

Computing a chain of isogenies

Given E and $x(P)$ for a generator P of G, compute φ_i's:

E

$x(P)$

Computing a chain of isogenies

Given E and $x(P)$ for a generator P of G, compute φ_i's:

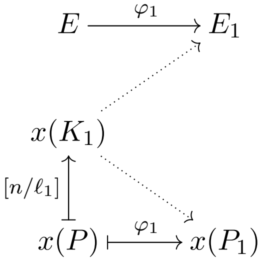
$$E$$

$$x(K_1)$$

$$\uparrow [n/\ell_1]$$

$$x(P)$$

Given E and $x(P)$ for a generator P of G, compute φ_i's:



Computing a chain of isogenies

Given E and $x(P)$ for a generator P of G, compute φ_i's:

$$E \xrightarrow{\varphi_1} E_1$$

$$\begin{array}{ccc} x(K_1) & & x(K_2) \\ \uparrow{\scriptstyle [n/\ell_1]} & & \uparrow{\scriptstyle [n/(\ell_1\ell_2)]} \\ x(P) & \xmapsto{\varphi_1} & x(P_1) \end{array}$$

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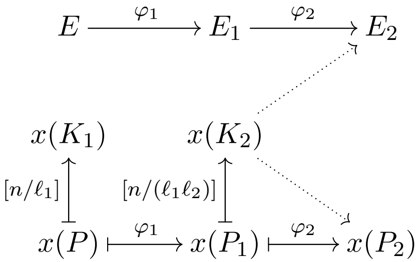

Computing a chain of isogenies

Given E and $x(P)$ for a generator P of G, compute φ_i's:

$$E \xrightarrow{\varphi_1} E_1 \xrightarrow{\varphi_2} E_2$$

$$\begin{array}{ccccc} x(K_1) & & x(K_2) & & x(K_3) \\ \uparrow{\scriptstyle [n/\ell_1]} & & \uparrow{\scriptstyle [n/(\ell_1\ell_2)]} & & \uparrow{\scriptstyle [n/(\ell_1\ell_2\ell_3)]} \\ x(P) & \xmapsto{\varphi_1} & x(P_1) & \xmapsto{\varphi_2} & x(P_2) \end{array}$$

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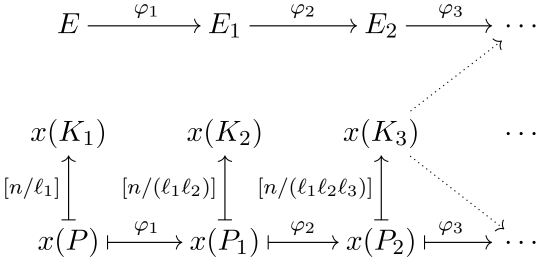

Computing a chain of isogenies

Given E and $x(P)$ for a generator P of G, compute φ_i's:

$$E \xrightarrow{\varphi_1} E_1 \xrightarrow{\varphi_2} E_2 \xrightarrow{\varphi_3} \cdots \xrightarrow{\varphi_{k-1}} E_{k-1} \xrightarrow{\varphi_k}$$

$$\begin{array}{ccccccccc}
x(K_1) & & x(K_2) & & x(K_3) & & \cdots & & \\
\uparrow [n/\ell_1] & & \uparrow [n/(\ell_1\ell_2)] & & \uparrow [n/(\ell_1\ell_2\ell_3)] & & & & \\
x(P) & \xmapsto{\varphi_1} & x(P_1) & \xmapsto{\varphi_2} & x(P_2) & \xmapsto{\varphi_3} & \cdots & \xmapsto{\varphi_{k-1}} & x(P_k)
\end{array}$$

Given E and $x(P)$ for a generator P of G, compute φ_i's:

$$
\begin{array}{ccccccccccc}
E & \xrightarrow{\varphi_1} & E_1 & \xrightarrow{\varphi_2} & E_2 & \xrightarrow{\varphi_3} & \cdots & \xrightarrow{\varphi_{k-1}} & E_{k-1} & \xrightarrow{\varphi_k} & E_k \\
 & & & & & & & & & \nearrow & \\
x(K_1) & & x(K_2) & & x(K_3) & & \cdots & & x(K_k) & & \\
{\scriptstyle [n/\ell_1]}\uparrow & & {\scriptstyle [n/(\ell_1\ell_2)]}\uparrow & & {\scriptstyle [n/(\ell_1\ell_2\ell_3)]}\uparrow & & & & \| & & \\
x(P) & \xmapsto{\varphi_1} & x(P_1) & \xmapsto{\varphi_2} & x(P_2) & \xmapsto{\varphi_3} & \cdots & \xmapsto{\varphi_{k-1}} & x(P_k) & &
\end{array}
$$

Cost of computing a chain of isogenies

We need to compute the following in each step:

- E_i : $O(\ell_i)$ operations by Vélu's formula.
- $x(P_i)$: $O(\ell_i)$ operations by Vélu's formula.
- $x(K_i)$: $O(\log(n/(\ell_1 \cdots \ell_i)))$ operations by binary multiplication.

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The total cost is

$$O\left(\sum_{i=1}^{k} \ell_i\right) + O\left(k \log(n) - \sum_{i=1}^{k} (k+1-i) \log(\ell_i)\right).$$

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Assume that $\max_i\{\ell_i\}$ in $O(1)$.

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Assume that $\max_i\{\ell_i\}$ in $O(1)$.

Then $k \in O(\log n)$, so the total cost is

$$O\left((\log n)^2\right).$$

Strategy

We can reduce the cost from

$$O((\log n)^2) \quad \text{to} \quad O(\log n \log\log n).$$

(so called *stragegy technique* proposed by [DFJP14])

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For simplicity, we assume that $n = \ell^k$.

We denote the cost of computing the following by

C_{cod} : the **codomain** of an ℓ-isogeny

C_{evl} : the **image of a point** under an ℓ-isogeny

C_{mul} : the **multiplication** by ℓ

Example of strategies

Let $P \in E$ be a point of order ℓ^3.
Decompose the separable isogeny with kernel $\langle P \rangle$ into

$$E \xrightarrow[\langle K_1 \rangle]{\varphi_1} E_1 \xrightarrow[\langle K_2 \rangle]{\varphi_2} E_2 \xrightarrow[\langle K_3 \rangle]{\varphi_3} E_3$$

Step	Objects	Cost
0	E, $x(P)$	
1	$x([\ell^2]P) = x(K_1)$	$(2\ C_{\mathsf{mul}})$
2	E_1, $x(\varphi_1(P))$	$(C_{\mathsf{cod}} + C_{\mathsf{evl}})$
3	$x([\ell]\varphi_1(P)) = x(K_2)$	(C_{mul})
4	E_2, $x(\varphi_1 \circ \varphi_2(P)) = x(K_3)$	$(C_{\mathsf{cod}} + C_{\mathsf{evl}})$
5	E_3	(C_{cod})

The total cost is $3C_{\mathsf{cod}} + 2C_{\mathsf{evl}} + 3C_{\mathsf{mul}}$.

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Step	Objects	Cost
0	E, $x(P)$	
1	$x([\ell]P), x([\ell^2]P) = x(K_1)$	$(2\ C_{\mathsf{mul}})$
2	E_1, $x(\varphi_1(P)), x(\varphi([\ell]P))$	$(C_{\mathsf{cod}} + 2C_{\mathsf{evl}})$
3	$x(\varphi_1([\ell]P)) = x([\ell]\varphi_1(P)) = x(K_2)$	(0)
4	E_2, $x(\varphi_1 \circ \varphi_2(P)) = x(K_3)$	$(C_{\mathsf{cod}} + C_{\mathsf{evl}})$
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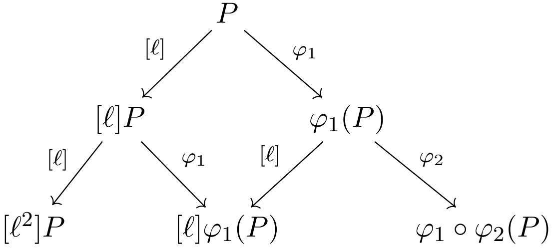
The total cost is $3C_{\mathsf{cod}} + 3C_{\mathsf{evl}} + 2C_{\mathsf{mul}}$.

$\Rightarrow$ We can replace C_{mul} by C_{evl}.

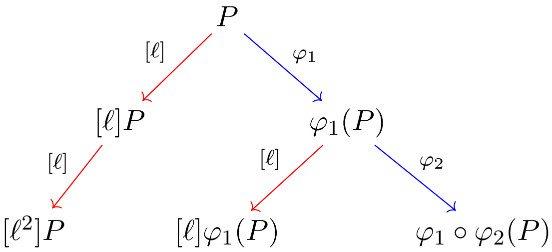
Visualization of strategies

The relationship among the points in the previous example:



Visualization of strategies

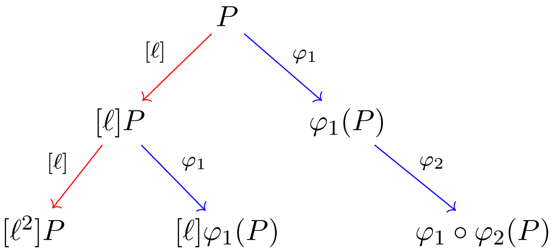
The first strategy:



The cost is $3C_{\mathsf{cod}} + 2C_{\mathsf{evl}} + 3C_{\mathsf{mul}}$.

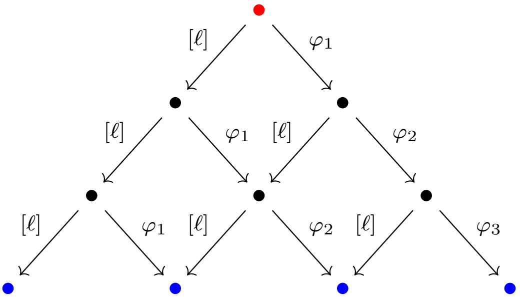
Visualization of strategies

The second strategy:



The cost is $3C_{\mathsf{cod}} + 3C_{\mathsf{evl}} + 2C_{\mathsf{mul}}$.

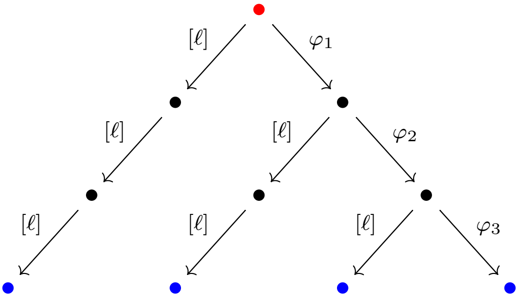
$k = 4$



Problem:

Choose edges connecting the top and bottom vertices to **minimize the cost**.

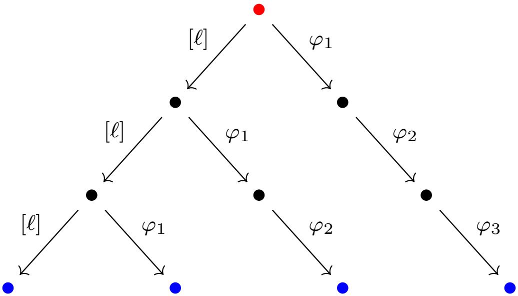
$k = 4$



Cost: $4C_{\text{cod}} + 3C_{\text{evl}} + 6C_{\text{mul}}$

We call this *multiplication-based strategy*.

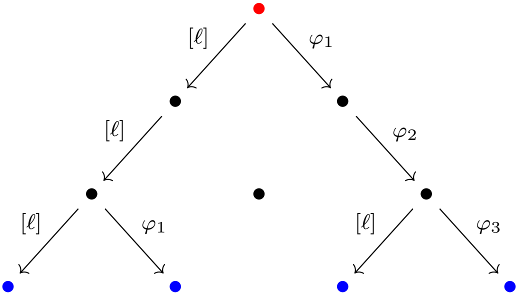
$k = 4$



Cost: $4C_{\mathsf{cod}} + 6C_{\mathsf{evl}} + 3C_{\mathsf{mul}}$

We call this *isogeny-based strategy*.

$k = 4$



Cost: $4C_{\mathsf{cod}} + 4C_{\mathsf{evl}} + 4C_{\mathsf{mul}}$

This strategy minimizes the cost if

$$\frac{1}{2}C_{\mathsf{mul}} \leq C_{\mathsf{evl}} \leq 2C_{\mathsf{mul}}.$$

Cost of strategy

Consider a chain of isogenies of length k.

The cost of the **multiplication-based strategy** is

$$kC_{\mathsf{cod}} + (k-1)C_{\mathsf{evl}} + \frac{k(k-1)}{2}C_{\mathsf{mul}}.$$

The cost of the **isogeny-based strategy** is

$$kC_{\mathsf{cod}} + \frac{k(k-1)}{2}C_{\mathsf{evl}} + (k-1)C_{\mathsf{mul}}.$$

These are $O(k^2)$.

Cost of optimized strategy

A strategy is *optimized* if its cost is **minimum** among all strategies of the same length.

We denote the cost of an **optimized strategy** by $C_{\mathsf{opt}}(k)$.

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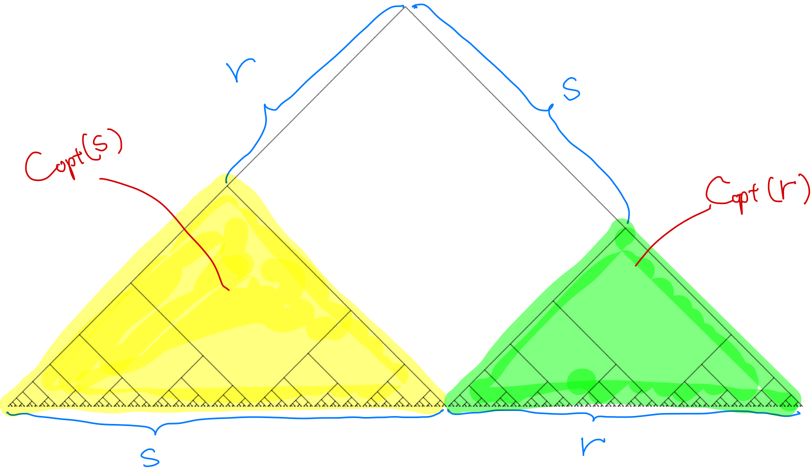
Theorem 30 (Lemma 4.5 in [DFJP14])

$$C_{\mathsf{opt}}(k) = \min_{r+s=k} \left\{ r \cdot C_{\mathsf{mul}} + s \cdot C_{\mathsf{evl}} + C_{\mathsf{opt}}(r) + C_{\mathsf{opt}}(s) \right\}.$$

Sketch of proof



* The figure is from [DFJP14].

Bound of cost

Theorem 31

Let $C = \max\{C_{\mathsf{evl}}, C_{\mathsf{mul}}\}$. *Then*

$$C_{\mathsf{opt}}(k) \leq k \cdot C_{\mathsf{cod}} + (k\lceil \log_2 k \rceil)C.$$

Sketch of proof

$$C_{\mathsf{opt}}(k) \leq \lfloor k/2 \rfloor C + \lceil k/2 \rceil C + C_{\mathsf{opt}}(\lfloor k/2 \rfloor) + C_{\mathsf{opt}}(\lceil k/2 \rceil)$$
$$= kC + C_{\mathsf{opt}}(\lfloor k/2 \rfloor) + C_{\mathsf{opt}}(\lceil k/2 \rceil)$$

Sketch of proof

$$
\begin{aligned}
C_{\mathsf{opt}}(k) &\le \lfloor k/2 \rfloor C + \lceil k/2 \rceil C + C_{\mathsf{opt}}(\lfloor k/2 \rfloor) + C_{\mathsf{opt}}(\lceil k/2 \rceil) \\
&= kC + C_{\mathsf{opt}}(\lfloor k/2 \rfloor) + C_{\mathsf{opt}}(\lceil k/2 \rceil) \\
&\le kC + kC + C_{\mathsf{opt}}(\lfloor \lfloor k/2 \rfloor /2 \rfloor) + C_{\mathsf{opt}}(\lceil \lfloor k/2 \rfloor /2 \rceil) \\
&\quad + C_{\mathsf{opt}}(\lfloor \lceil k/2 \rceil /2 \rfloor) + C_{\mathsf{opt}}(\lceil \lceil k/2 \rceil /2 \rceil) \\
&= 2kC + C_{\mathsf{opt}}(\lfloor \lfloor k/2 \rfloor /2 \rfloor) + C_{\mathsf{opt}}(\lceil \lfloor k/2 \rfloor /2 \rceil) \\
&\quad + C_{\mathsf{opt}}(\lfloor \lceil k/2 \rceil /2 \rfloor) + C_{\mathsf{opt}}(\lceil \lceil k/2 \rceil /2 \rceil)
\end{aligned}
$$

Sketch of proof

$$
\begin{aligned}
C_{\mathsf{opt}}(k) &\leq \lfloor k/2 \rfloor C + \lceil k/2 \rceil C + C_{\mathsf{opt}}(\lfloor k/2 \rfloor) + C_{\mathsf{opt}}(\lceil k/2 \rceil) \\
&= kC + C_{\mathsf{opt}}(\lfloor k/2 \rfloor) + C_{\mathsf{opt}}(\lceil k/2 \rceil) \\
&\leq kC + kC + C_{\mathsf{opt}}(\lfloor \lfloor k/2 \rfloor /2 \rfloor) + C_{\mathsf{opt}}(\lceil \lfloor k/2 \rfloor /2 \rceil) \\
&\quad + C_{\mathsf{opt}}(\lfloor \lceil k/2 \rceil /2 \rfloor) + C_{\mathsf{opt}}(\lceil \lceil k/2 \rceil /2 \rceil) \\
&= 2kC + C_{\mathsf{opt}}(\lfloor \lfloor k/2 \rfloor /2 \rfloor) + C_{\mathsf{opt}}(\lceil \lfloor k/2 \rfloor /2 \rceil) \\
&\quad + C_{\mathsf{opt}}(\lfloor \lceil k/2 \rceil /2 \rfloor) + C_{\mathsf{opt}}(\lceil \lceil k/2 \rceil /2 \rceil) \\
&\;\vdots \\
&\leq k \lceil \log_2 k \rceil C + kC_{\mathsf{opt}}(1) \\
&= k \lceil \log_2 k \rceil C + kC_{\mathsf{cod}}.
\end{aligned}
$$

Example

Assume $k = 100$ and $C_{\mathsf{cod}} = C_{\mathsf{evl}} = C_{\mathsf{mul}} = C$.

The cost the mulitplication-based (isogeny-based) strategy is

$$100C + 99C + 4950C = 5149C.$$

The cost of the optimized strategy is

$$100C + 672C = 772C.$$

This is about 15% of the cost of the multiplication-based strategy.

How to compute the optimized strategy?

There exists an algorithm to compute an optimized strategy.
(see Algorithm 60 in [JAC+22])

- Use Theorem 30.
- The computation is **recursive**.
- The cost is $O(k^2)$.
- In applications, an optimized strategy is computed in **advance**.
  $\because k$ is fixed (in most cases).

Further topics

Modular polynomials

Let $n > 1$ be an integer.

The *modular polynomial of order n* is a polynomial $\Phi_n(X, Y) \in \mathbb{Z}[X, Y]$ such that

$$\Phi_n(j_1, j_2) = 0 \Leftrightarrow \exists n\text{-isogeny } \varphi : E_{j_1} \to E_{j_2},$$

where E_{j_i} is the elliptic curve with j-invariant j_i.

Example:

$$\begin{aligned}\Phi_2(X,Y) = &X^3 + Y^3 - X^2Y^2 + 1488(X^2Y + XY^2) - 162000(X^2 + Y^2) \\ &+ 40773375XY + 8748000000(X + Y) - 157464000000000.\end{aligned}$$

See Chapter 10.3 in [Was08] or Chapter 5 in [Lan87] for more details.

$\sqrt{}$élu's formulas

A *$\sqrt{}$élu's formula* is an algorithm to compute an ℓ-isogeny.

- by [BDFLS20].
- based on Vélu's formula.
- The cost is $\tilde{O}(\sqrt{\ell})$ operations, not $O(\ell)$.
- uses the resultant of two polynomials.

In practice, $\sqrt{}$élu's formulas are faster than Vélu's formulas for $\ell > 100$.

Radical isogenies

Radical isogenies are formulas to compute an ℓ-isogeny.

- by [CDV20],
- uses an ℓ-th root (radical) of an element.

Which of Vélu's formulas or radical isogenies is faster depends on applications.

Appendix

Montgomery curves

Definition 32

A *Montgomery curve* is an elliptic curve defined by

$$E_A : y^2 = x^3 + Ax^2 + x, \quad A^2 \neq 4.$$

We call A the *Montgomery coefficient* of E_A.

We denote the Montgomery curve with coefficient A by E_A.

Addition on Montgomery curves

Proposition 33 (§10.3 in [Mon87])

Let E_A be a Montgomery curve with the Montgomery coefficient A, and $P, Q \in E_A \setminus \{0_{E_A}\}$. Then, the following hold:

$$x(P+Q)x(P-Q) = \left(\frac{x(P)x(Q)-1}{x(P)-x(Q)}\right)^2,$$

$$x(2P) = \frac{(x(P)^2-1)^2}{4(x(P)^3 + A \cdot x(P)^2 + x(P))}.$$

Note:

- $x(P) - x(Q) = 0 \Leftrightarrow P + Q = 0_{E_A}$ or $P - Q = 0_{E_A}$.
- $x(P)^3 + A \cdot x(P)^2 + x(P) = 0 \Leftrightarrow [2]P = 0_{E_A}$.

xADD and xDBL on Montgomery curves

Let E_A be a Montgomery curve and $P, Q \in E_A$.

From Proposition 33, we define the following two algorithms.

xADD:

Input: $A, x(P), x(Q), x(P - Q)$
Output: $x(P + Q)$

xDBL:

Input: $A, x(P)$
Output: $x([2]P)$

Scalar multiplication on Montgomery curves

Algorithm 1: Montgomery ladder

Input: A Montgomery coefficient A, the x-coordinate of a point $P \in E_A$, and an integer $n > 0$.

Output: The x-coordinate of $[n]P$.

Let $(n_0, n_1, \dots, n_k)$ be the binary expansion of n. // $n = \sum_{i=0}^{k} n_i 2^i$.
Let $(x_0, x_1) := (x(P), x([2]P))$
for $i = k - 1$ **to** 0 **do**
  if $n_i = 1$ **then**
    $(x_0, x_1) := (\mathsf{xADD}(A, x_0, x_1, x(P)), \mathsf{xDBL}(A, x_0))$
  else
    $(x_0, x_1) := (\mathsf{xDBL}(A, x_0)), \mathsf{xADD}(A, x_0, x_1, x(P))$
  // $x_0 = x([n_k 2^{k-i} + \cdots + n_i 2^i]P)$
  // $x_1 = x([n_k 2^{k-i} + \cdots + n_i 2^i + 1]P)$
return x_0

Remarks on Montgomery ladder

- We can give a constant-time implementation of the Montgomery ladder.

  I.e., the computational time only depends on the bit-length of the scalar n, not on the value of n.

- If we do not need a constant-time implementation, we can construct a more efficient *differential addition chain* (see [CS17] for more details).

Theorem 34 (2-isogeny formula, Section 4.3 in [JD11])

An isogeny $\varphi : E_A \to E_{A'}$ *with kernel* $\langle (0,0) \rangle$ *is given by*

$$A' = \frac{A+6}{2\sqrt{A+2}},$$

$$x(\varphi(P)) = \frac{(x(P)-1)^2}{(2\sqrt{A+2})x(P)} \text{ for } P \in E_A.$$

Isogeny formulas on Montgomery curves

Theorem 35 (2-isogeny formula, Section 1.1.9 in [JAC+22])

Let $(x_2, 0)$ *be a point on* E_A *of order 2.*
Then an isogeny $\varphi : E_A \to E_{A'}$ *with kernel* $\langle (x_2, 0) \rangle$ *is given by*

$$A' = 2(2 - x_2),$$

$$x(\varphi(P)) = \frac{x(P)(x_2 - x(P))}{x(P) - x_2} \text{ for } P \in E_A.$$

Isogeny formulas on Montgomery curves

Theorem 36 (4-isogeny formula, Section 4.3.2 in [DFJP14])

An isogeny $\varphi : E_A \to E_{A'}$ *with kernel* $\langle(1, \sqrt{A+2})\rangle$ *is given by*

$$A' = 2\frac{A+6}{A-2},$$

$$x(\varphi(P)) = \frac{(x(P)+1)^2(x(P)^2 + Ax(P) + 1)}{(2-A)x(P)(x(P)-1)^2} \text{ for } P \in E_A.$$

Theorem 37 (4-isogeny formula, Section 4.3.2 in [DFJP14])

An isogeny $\varphi : E_A \to E_{A'}$ *with kernel* $\langle(-1, \sqrt{A-2})\rangle$ *is given by*

$$A' = -2\frac{A-6}{A+2},$$

$$x(\varphi(P)) = -\frac{(x(P)-1)^2(x(P)^2 + Ax(P) + 1)}{(2+A)x(P)(x(P)+1)^2} \text{ for } P \in E_A.$$

Isogeny formulas on Montgomery curves

Theorem 38 (4-isogeny formula, Section 1.1.9 in [JAC+22])

Let (x_4, y_4) *be a point on* E_A *of order 4.*
Then an isogeny $\varphi : E_A \to E_{A'}$ *with kernel* $\langle (x_4, y_4) \rangle$ *is given by*

$$A' = 4x_4^4 - 2,$$

$$x(\varphi(P)) = -\frac{x(P)((x_4^2+1)x(P) - 2x_4)(x_4 x(P) - 1)^2}{(x(P) - x_4)^2(2x_4 x(P) - x_4^2 - 1)} \text{ for } P \in E_A.$$

Theorem 39 (Odd-degree isogeny formula, Theorem 1 in [CH17])

Let K be a point on E_A of odd order ℓ. We denote the x-coordinate of $[i]K$ by x_i for $i = 1, 2, \ldots, (\ell-1)/2$.
Then an isogeny $\varphi : E_A \to E_{A'}$ with kernel $\langle K \rangle$ is given by

$$A' = \left(6 \sum_{i=1}^{\frac{\ell-1}{2}} \left(\frac{1}{x_i} - x_i \right) + A \right) \left(\prod_{i=1}^{\frac{\ell-1}{2}} x_i \right)^2,$$

$$x(\varphi(P)) = x(P) \left(\prod_{i=1}^{\frac{\ell-1}{2}} \frac{x_i x(P) - 1}{x(P) - x_i} \right)^2.$$

Isogeny formulas on Montgomery curves

Theorem 40 (Odd-degree isogeny formula, Section 4.2 in [MR18])

We use the same notation as in the previous theorem. Then we have

$$A' = 2\frac{a+d}{a-d},$$

where a and d are defined by

$$a = (A+2)^{\ell}\left(\prod_{i=1}^{\frac{\ell-1}{2}}(x_i+1)\right)^8,$$

$$d = (A-2)^{\ell}\left(\prod_{i=1}^{\frac{\ell-1}{2}}(x_i-1)\right)^8.$$

Note: This formula is more efficient than the previous one if $\ell \geq 7$.

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